Super-reflexive Banach spaces and power growth of certain operators

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Let X be a reflexive Banach space. Let $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be a spectral family in X. We define its q-variation, for each $1 < q < \infty$ as

$$var_q(E) = \sup_{\|x\|\leq 1} \sup_{a>0} \sup_{u\in\mathcal{P}_{[-a,a]}} \Big(\sum_{j=1}^{N} \|E(\lambda_j)x - E(\lambda_{j-1})x\|^q\Big)^{\frac{1}{q}}$$

Here $\mathcal{P}_{[-a,a]}$ denotes the set of partitions $u = \{-a = \lambda_0, ..., \lambda_N = a\}$ of the interval $[-a, a]$.

We say that $T \in \mathcal{B}(X)$ is trigonometrically well-bounded if it has a Riemann-Stieltjes integral representation $T = \int_{0^-}^{2\pi} e^{i\lambda}\, dE(\lambda)$ in terms of a spectral family $E(\lambda)$. Any spectral family is automatically uniformly bounded, i.e. $\sup_\lambda \|E(\lambda)\| < \infty$. Further, if T is trigonometrically well-bounded, it is easy to show that T is invertible and that an upper bound for the power growth of T is $|n|$, i.e. $\|T^n\| \leq K_T|n|$ for some $K_T > 0$. Here we improve on both these results as follows, by restricting ourselves to super-reflexive Banach spaces.

Theorem 1. *Let X be a super-reflexive Banach space. Then there exists $1 \leq q_E < \infty$ such that $var_{q_E}(E) < \infty$.*

Theorem 2. *Let X be a super-reflexive Banach space and $T \in \mathcal{B}(X)$ an invertible trigonometrically well-bounded operator. Then there exists a number $1 < \alpha_{X,E}$ and a constant $C_{X,E} > 0$ such that*

$$\|T^n\| \leq C_{X,E}|n|^{\alpha_{X,E}} \quad \textit{for all } n \in \mathbb{Z}\setminus 0\,.$$

Theorem 2 is clearly an improvement on the previous power growth estimate. We briefly examine examples of trigonometrically well-bounded operators on the weighted scalar $l_p(\mathbb{Z})$ spaces. From these we deduce that if $X = L^p(\mu)$, and T is an arbitrary invertible trigonometrically well-bounded operator, the slowest general power growth we may hope to establish is $|n|^{1/p}$.