

Direct numerical simulation of the evolution of high-order statistical momenta of weakly nonlinear water wave field

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Workshop on Rogue Waves, ICMS, Edinburgh, 12-15 December 2005

Motivation

Existing statistical approach to water waves: spectral transfer (Hasselmann, or kinetic) equation for water waves. Wave field is assumed to be quasi-Gaussian.

A verification method based on **direct numerical simulation** is an open problem.

- Independent verification of the fundamental hypotheses
- Higher order nonlinear processes cannot be included
- Does not allow to trace the evolution of higher momenta
- Observations of long-lived coherent patterns contradict the applicability of the approach

Direct numerical simulation

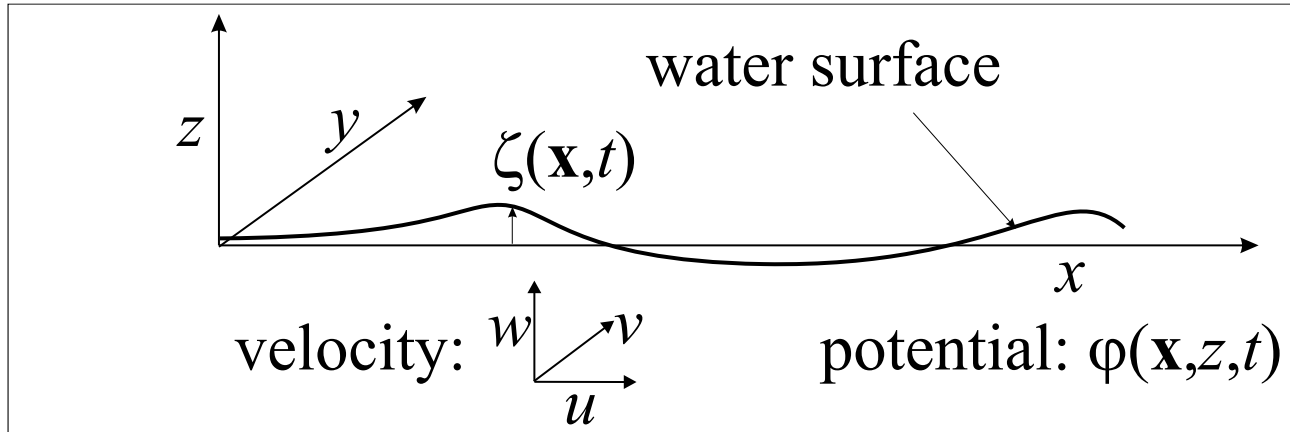
- Is it possible in principle? Can **quantitative** comparison be achieved? How a **continuous** wave field can be represented numerically?

There is understanding that

(Janssen 2003; Annenkov & Shrira 2005, submitted to JFM)

- Approximately resonant interactions must be taken into account (exact resonances are not essential)
- At the initial stage, evolution has fast $O(\varepsilon^{-2})$ timescale
- To avoid ‘frozen turbulence’, a very large number of interactions is required

Formulation



3D potential gravity waves on the free surface of an incompressible fluid. Wave slopes $O(\varepsilon)$.

$$\frac{\partial \zeta(\mathbf{x}, t)}{\partial t} = \frac{\delta H}{\delta \psi(\mathbf{x}, t)}, \quad \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = - \frac{\delta H}{\delta \zeta(\mathbf{x}, t)}$$

where $\psi(\mathbf{x}, t) = \varphi(\mathbf{x}, \zeta(\mathbf{x}, t), t)$, and the Hamiltonian H is the total energy of the system.

Reduced (Zakharov) equation

$$i \frac{\partial B_0}{\partial t} = \int T_{0123} B_1^* B_2 B_3 e^{i\Delta\omega t} \delta_{0+1-2-3} d\mathbf{k}_{123}$$

Compact notation: $B_0 = B(\mathbf{k}_0)$, $\delta_{0+1-2-3} = \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, etc.

Nonlinear normal variables: $B(\mathbf{k}, t) = b(\mathbf{k}, t) \exp [i\omega(\mathbf{k})t]$,

$$b(\mathbf{k}) = \frac{1}{\sqrt{2}} \left\{ \sqrt{\frac{\omega(\mathbf{k})}{k}} \zeta(\mathbf{k}) + i \sqrt{\frac{k}{\omega(\mathbf{k})}} \psi(\mathbf{k}) \right\} + O(\varepsilon),$$

Kinetic (Hasselmann) equation

$$\frac{\partial n_0}{\partial t} = 4\pi \int T_{0123}^2 f_{0123} \delta_{0+1-2-3} \delta(\Delta\omega) d\mathbf{k}_{123}$$

$$f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3), \quad \langle b_0^* b_1 \rangle = n_0 \delta_{0-1}$$

Direct numerical simulation - how?

Using a regular grid?

Lots of interactions, most of them non-resonant, not even approximately. Waste of computational resources, frozen turbulence, for a grid of any size. Impossible to represent higher-order nonlinear processes. Undesirable artefacts (Kartashova 1998).

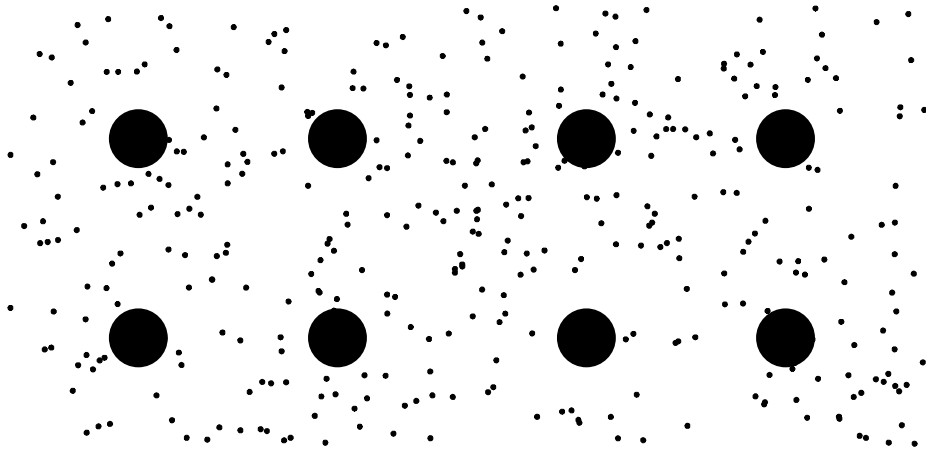
Using an arbitrary grid?

Too few interactions, effects of discreteness inevitable. Frozen turbulence, for a grid of any size.

How to represent a continuous field then?

Suppose that we have N discrete harmonics. We then have $N(N + 1)/2$ degenerated ('elastic') interactions ($\mathbf{k}_i + \mathbf{k}_j = \mathbf{k}_i + \mathbf{k}_j$), and M resonant interactions. We need $M = O(N^2)$. A discrete grid cannot provide this!

Continuous field



The question is: **what is more important, rigorous discretization, or continuous nature of interactions?**

Around each grid point, there are lots of other resonances, which do not fit into grid points exactly, but represent **approximate** interactions between grid points.

Alternative formulation: let's consider interaction of finite-width **wave packets**, each having **one** amplitude and phase, but linked by a rich system of interactions.

Parameters

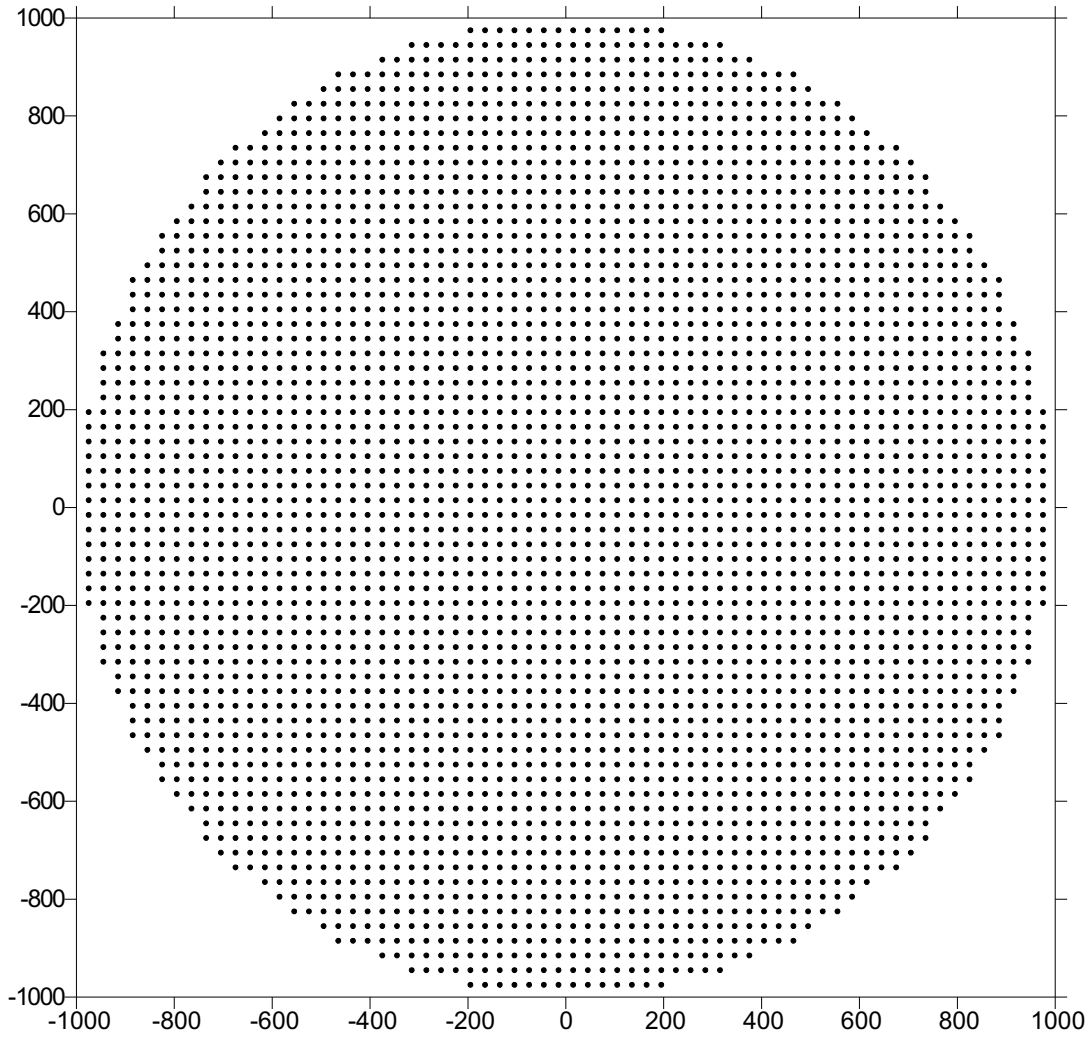
We have actually two different grids:

- grid of N wavepackets
- grid of M resonances linking them

Computational effort is proportional to N^2 and M (so it makes sense to choose $M \sim N^2$).

Numerical justification: there must be no dependence on the choice of M .

A circular grid



$$N = 3.5 \cdot 10^3, M = O(10^7)$$

Questions

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- Are the integrals conserved?
Yes. We are actually modifying the Hamiltonian.
- Aren't we inserting gaussianity in this way?

Kinetic equation properties

Direct and inverse cascades in wind-driven sea

21

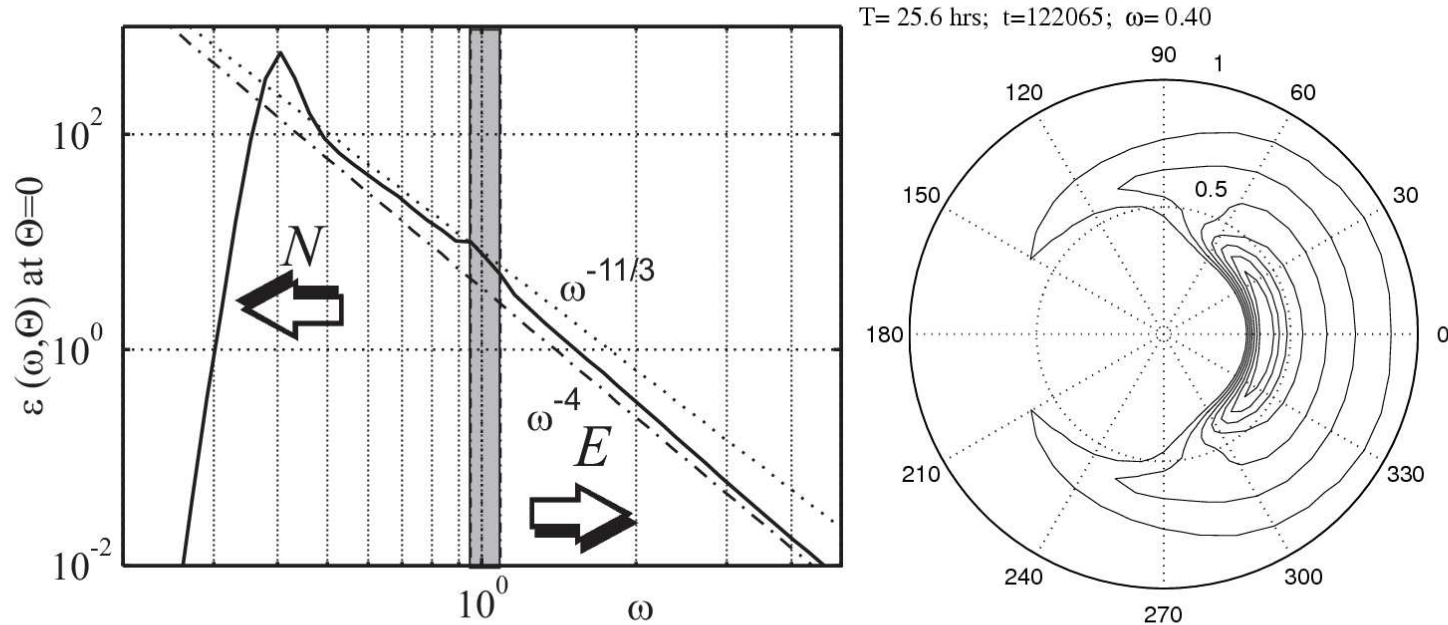
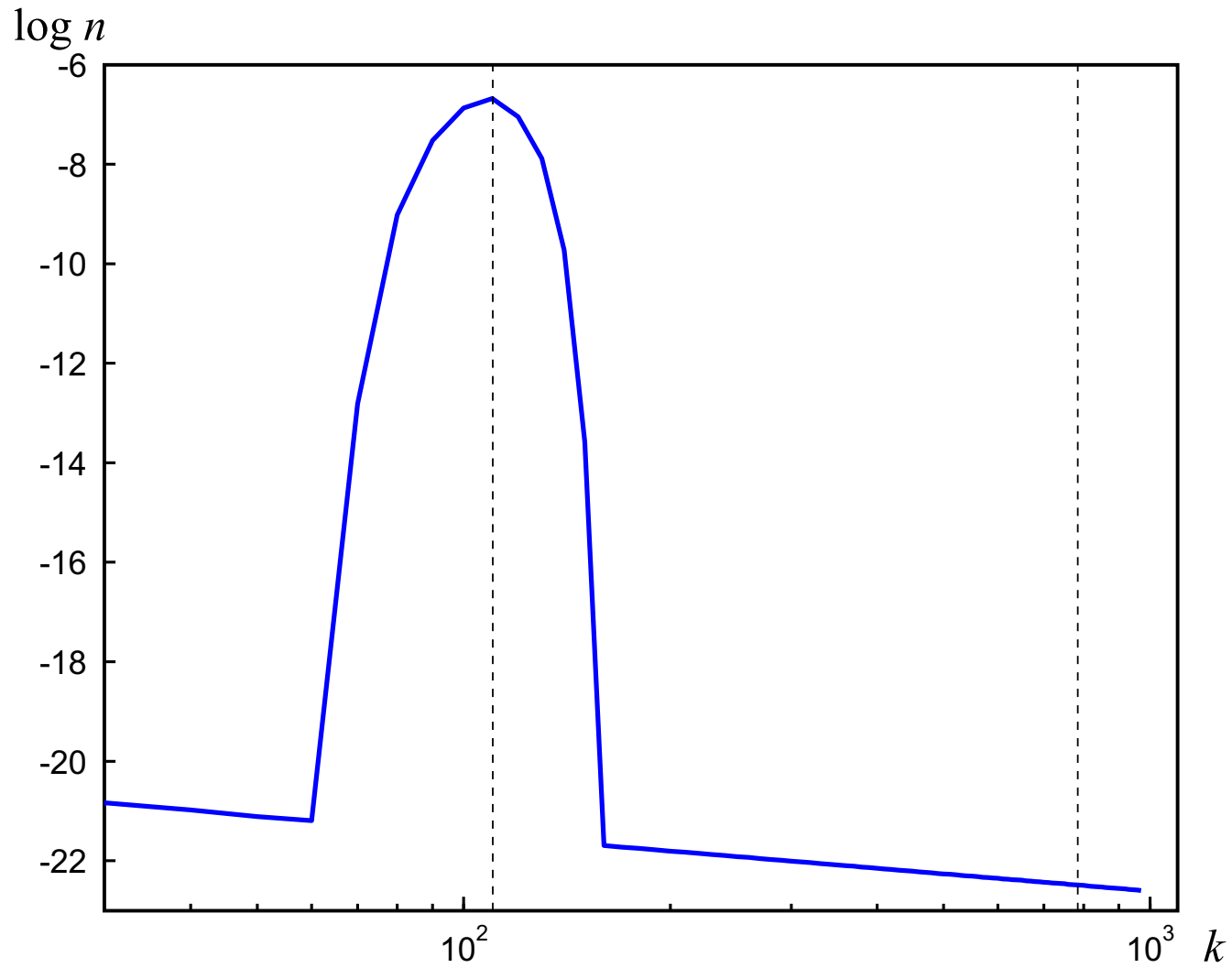


FIGURE 5. Form of the spectra and its directional distribution for ‘artificial’ source function. The solution is tending to Kolmogorov’s asymptotics, i.e. to the inverse cascade regime in low frequency and to the direct cascade in high frequencies

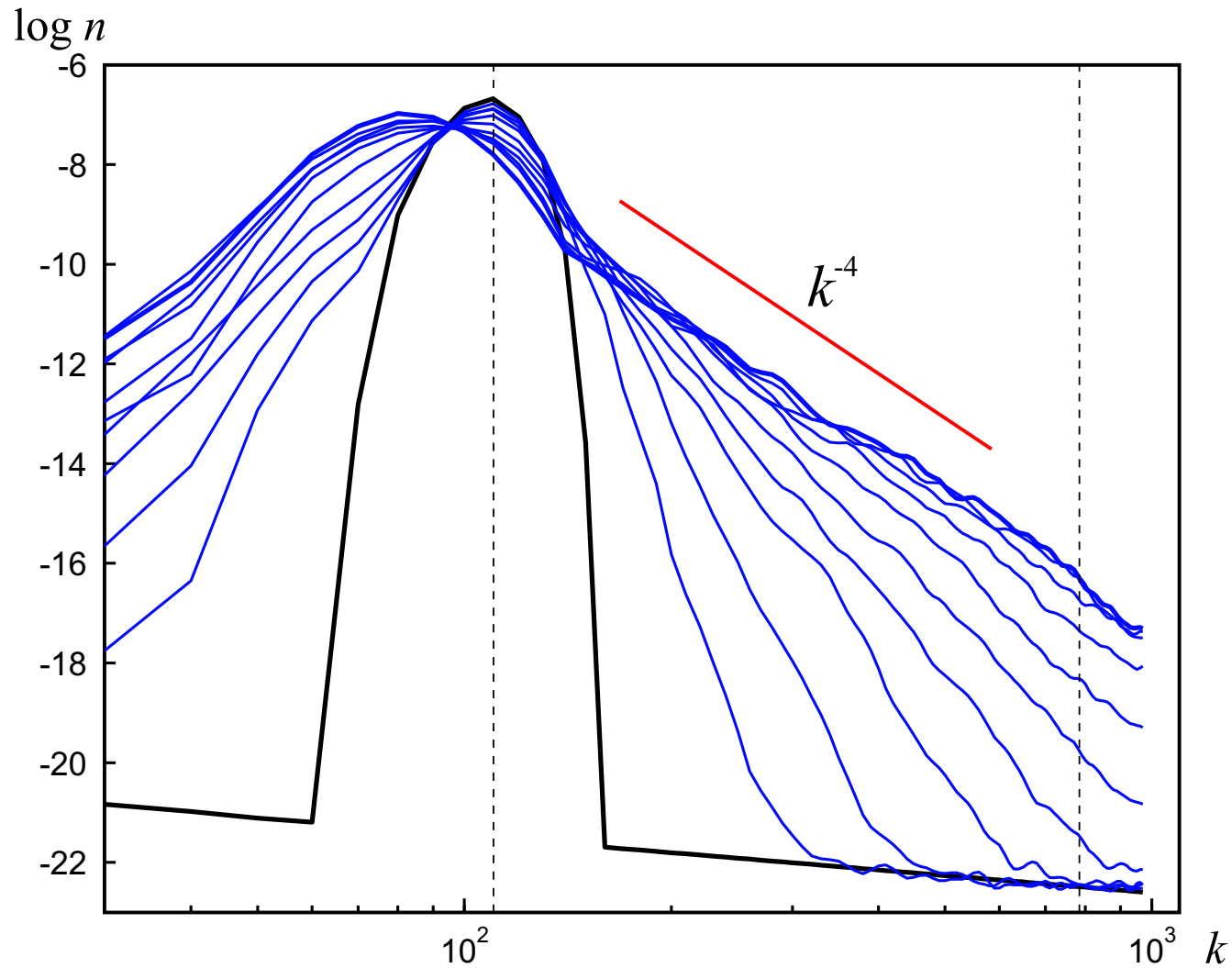
4. Self-similar solutions for the kinetic equation

The Kolmogorov solutions presented above are closely related with scaling invariance of the kinetic equation (2.31) for the case of deep water waves. This invariance can be used for deeper insight in wave kinetics and construction of self-similar solutions for non-stationary kinetic equation. The found non-stationary self-similar solutions can be related to the stationary Kolmogorov solutions and, thus, the Kolmogorov solutions can be treated not only as a mathematical object but as a physically relevant state of wave

First example: direct cascade

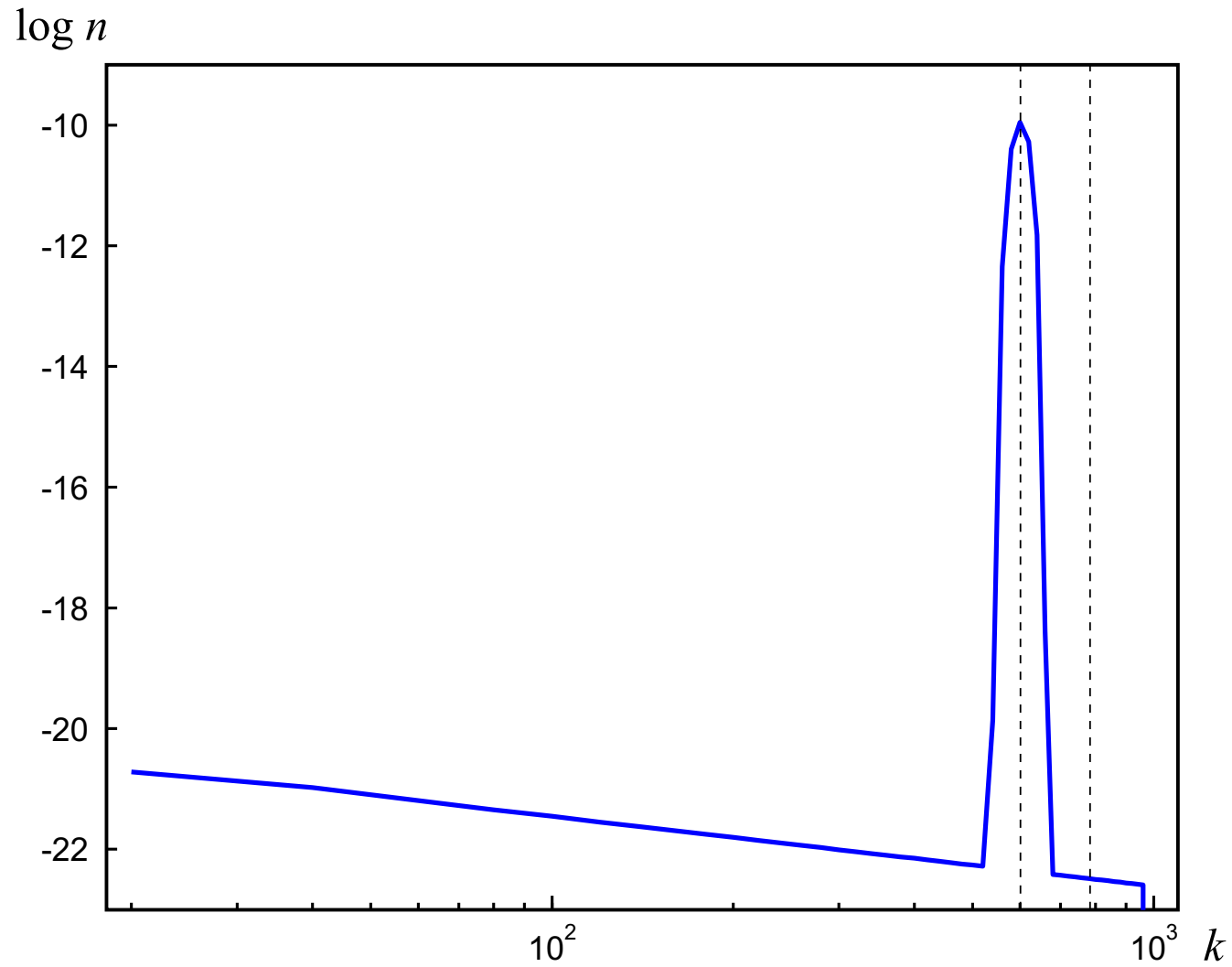


First example: direct cascade

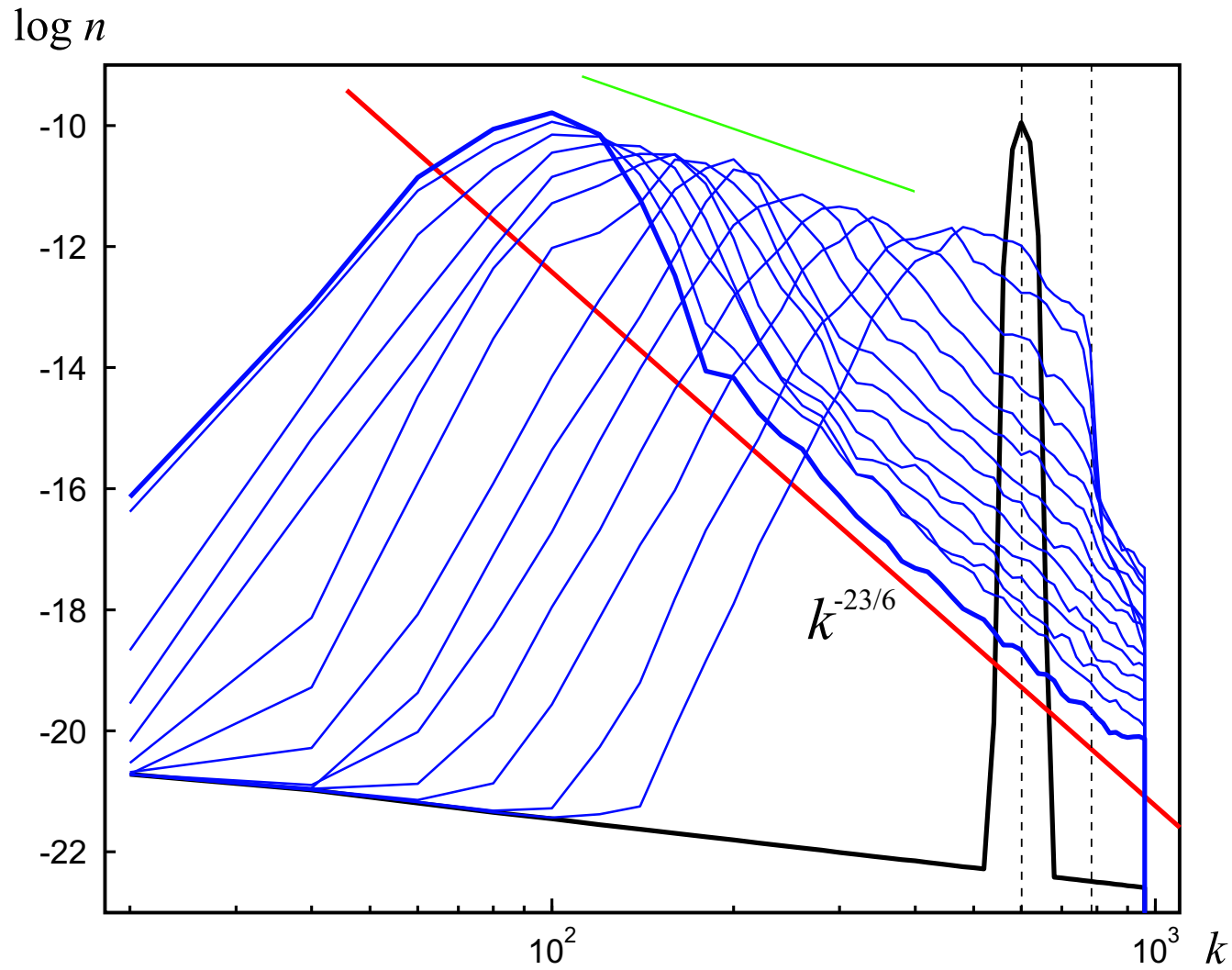


$$T = 20m, m = 0 \dots 10$$

Inverse cascade (dissipation only)



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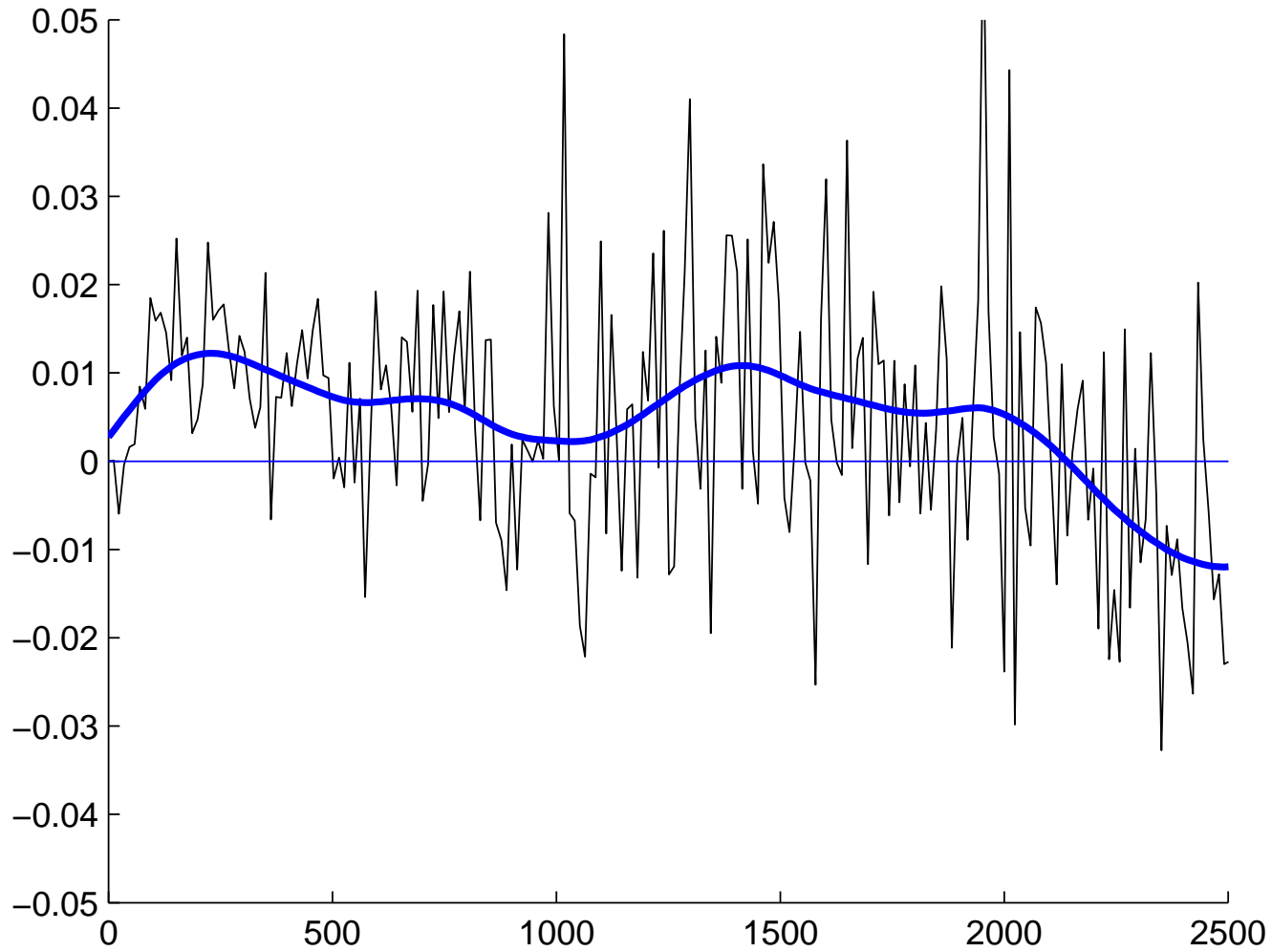


$$T = 20 \cdot 2^m, m = 0 \dots 13$$

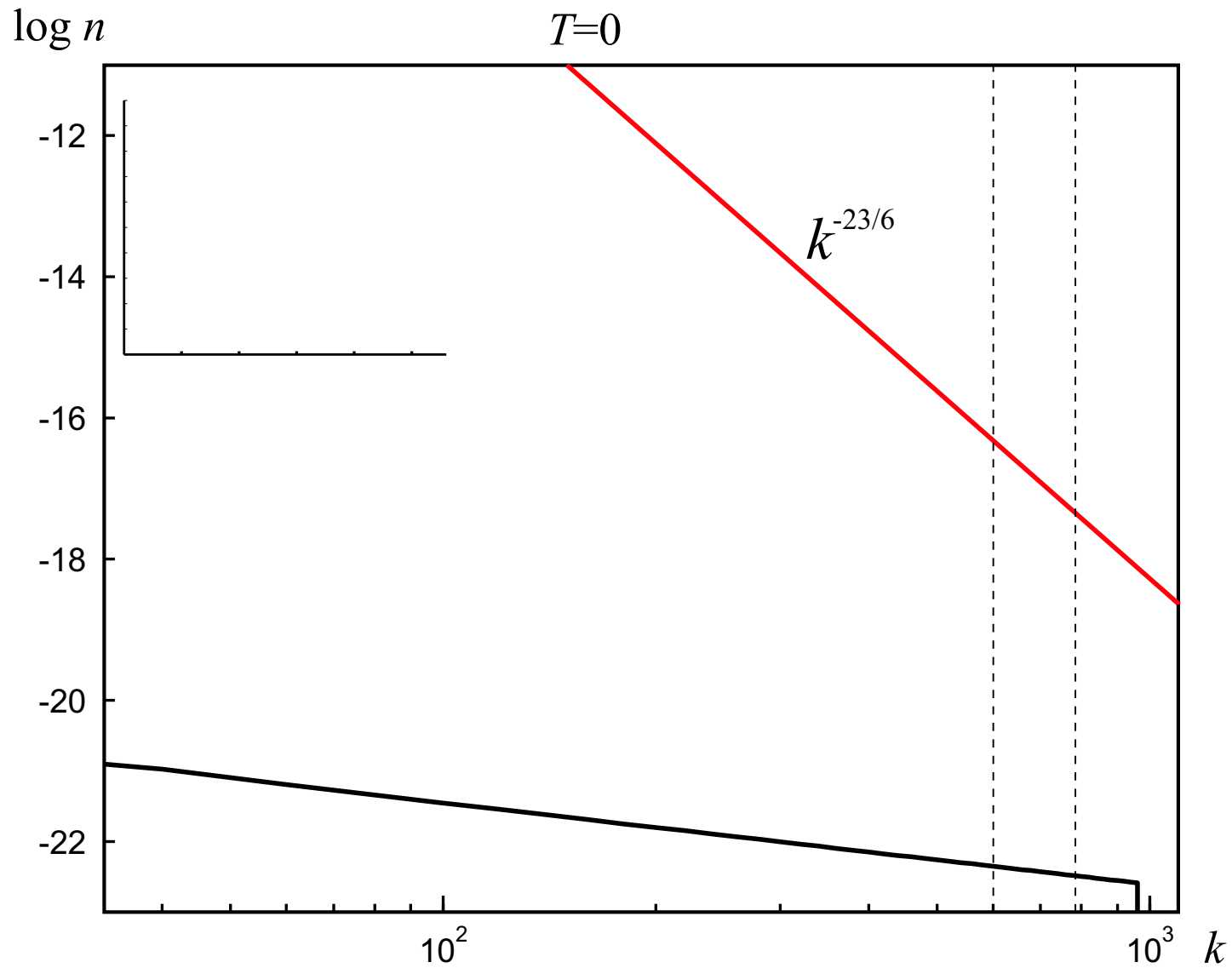
Inverse cascade (intermittency)



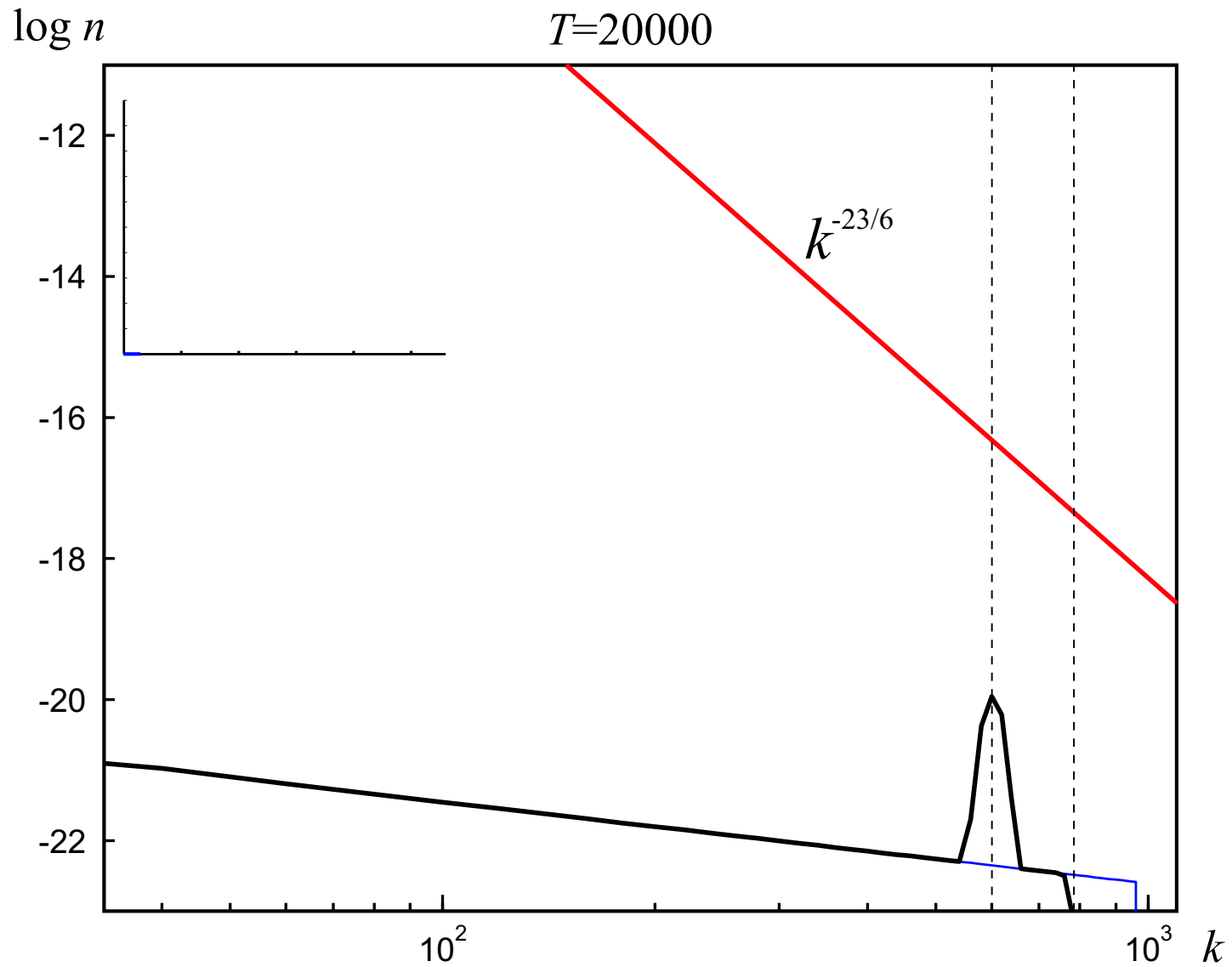
Inverse cascade (kurtosis)



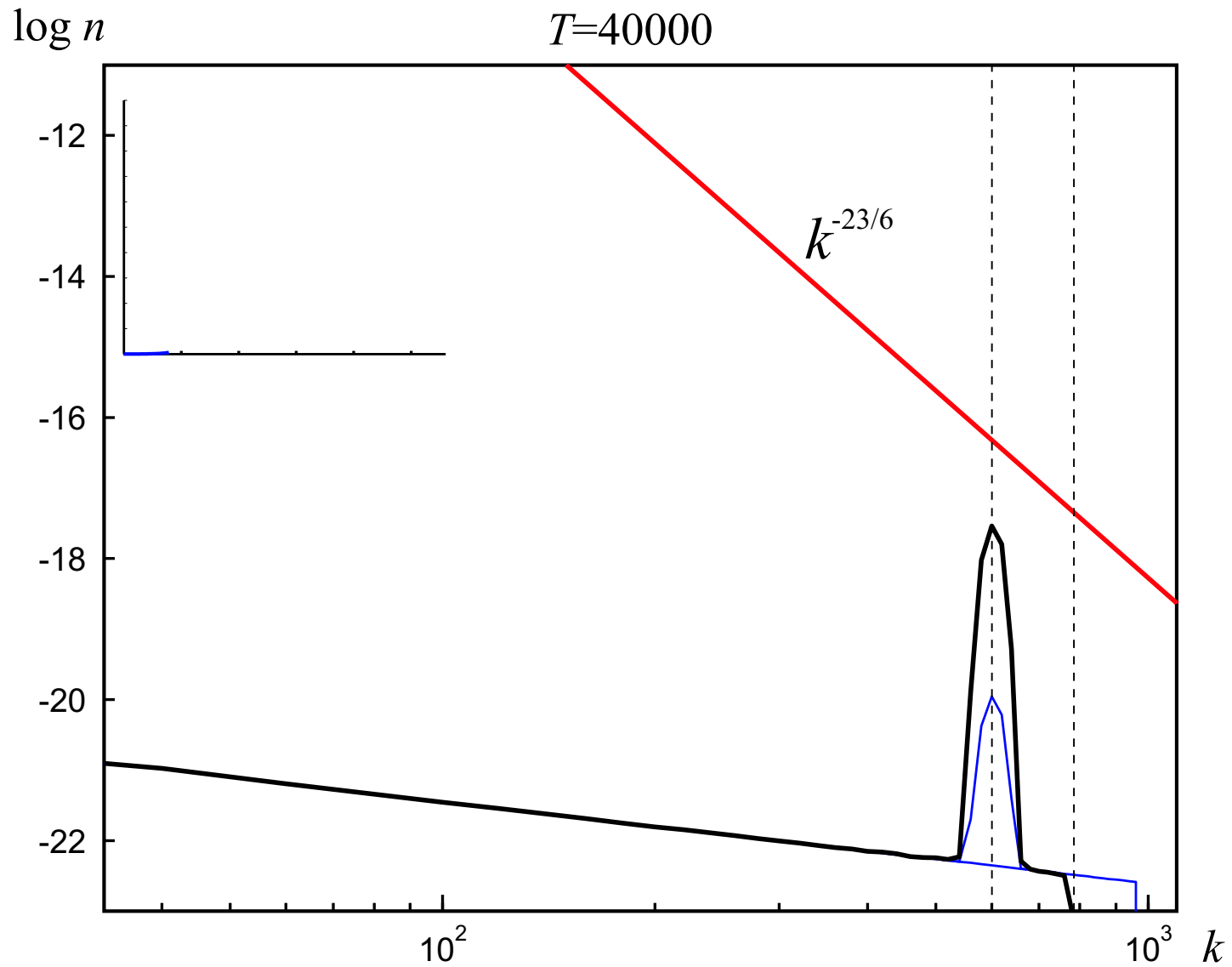
Forced inverse cascade



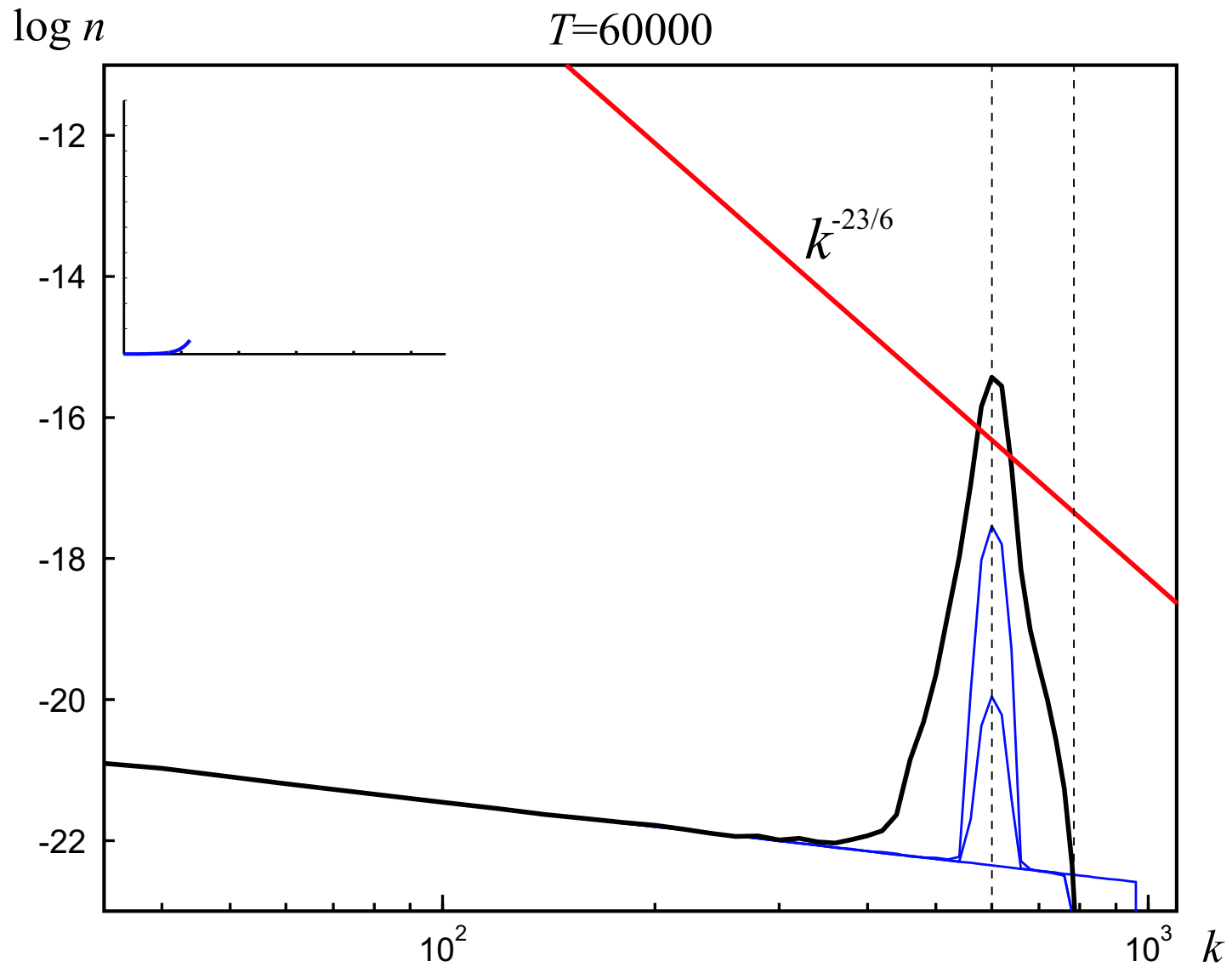
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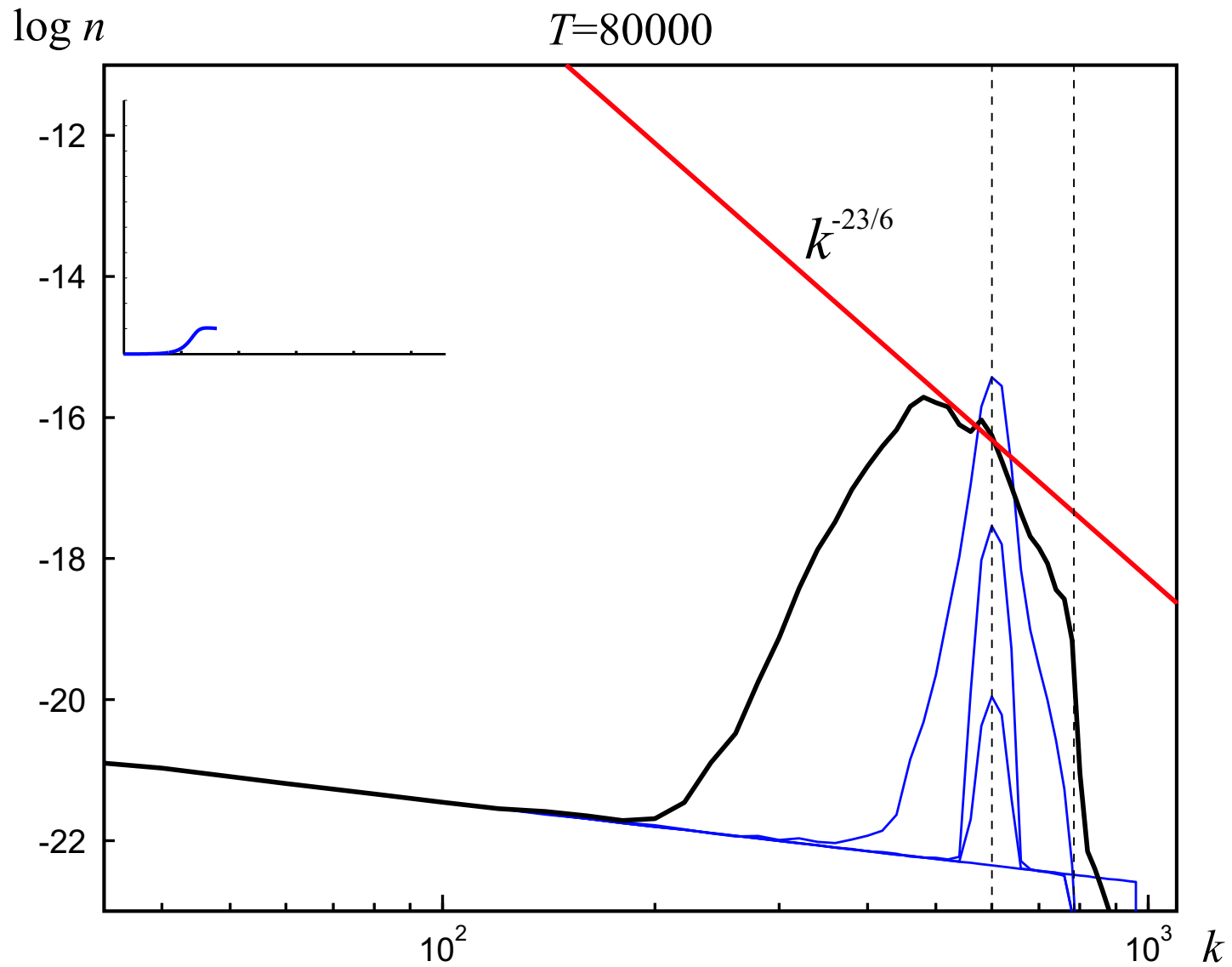
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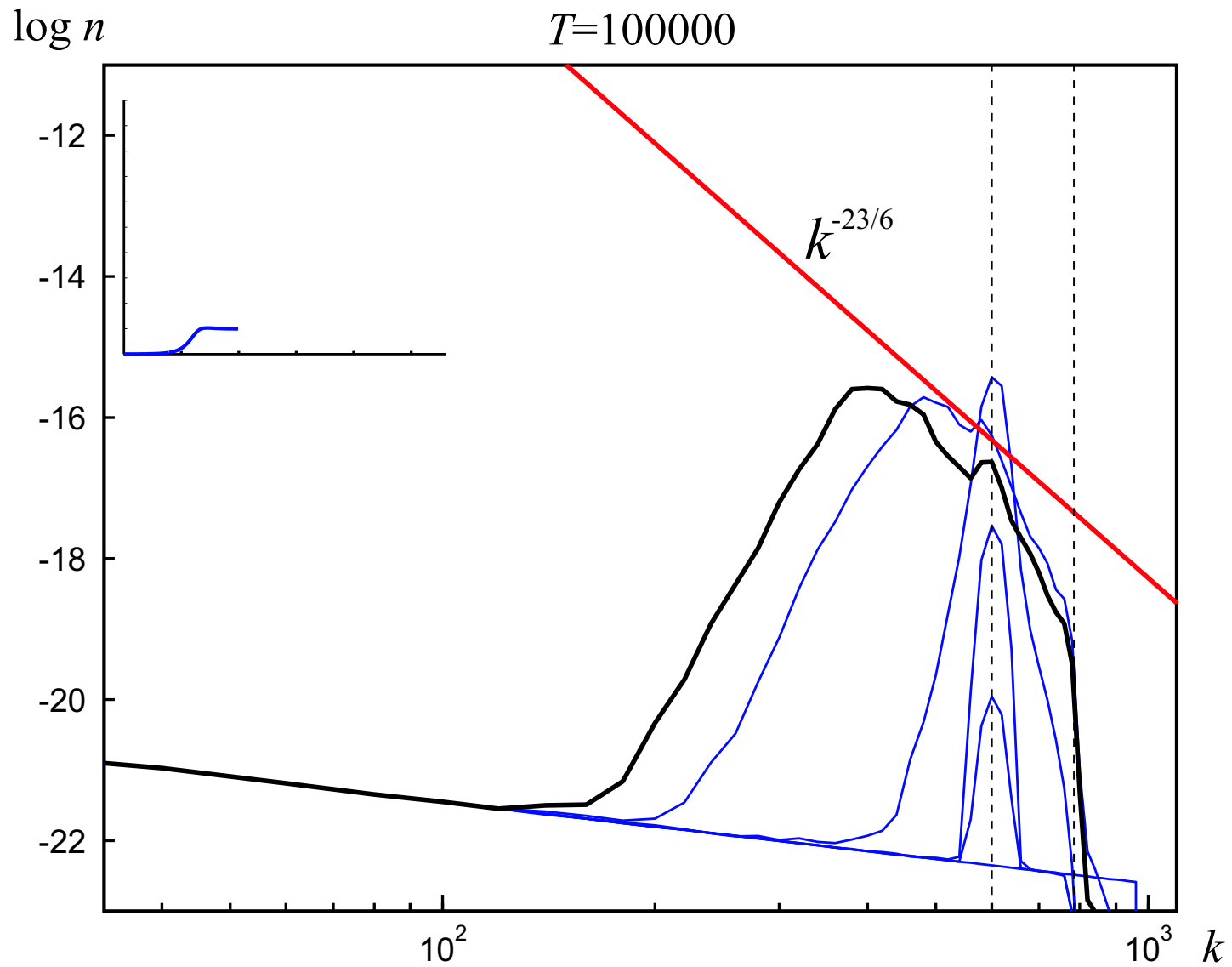
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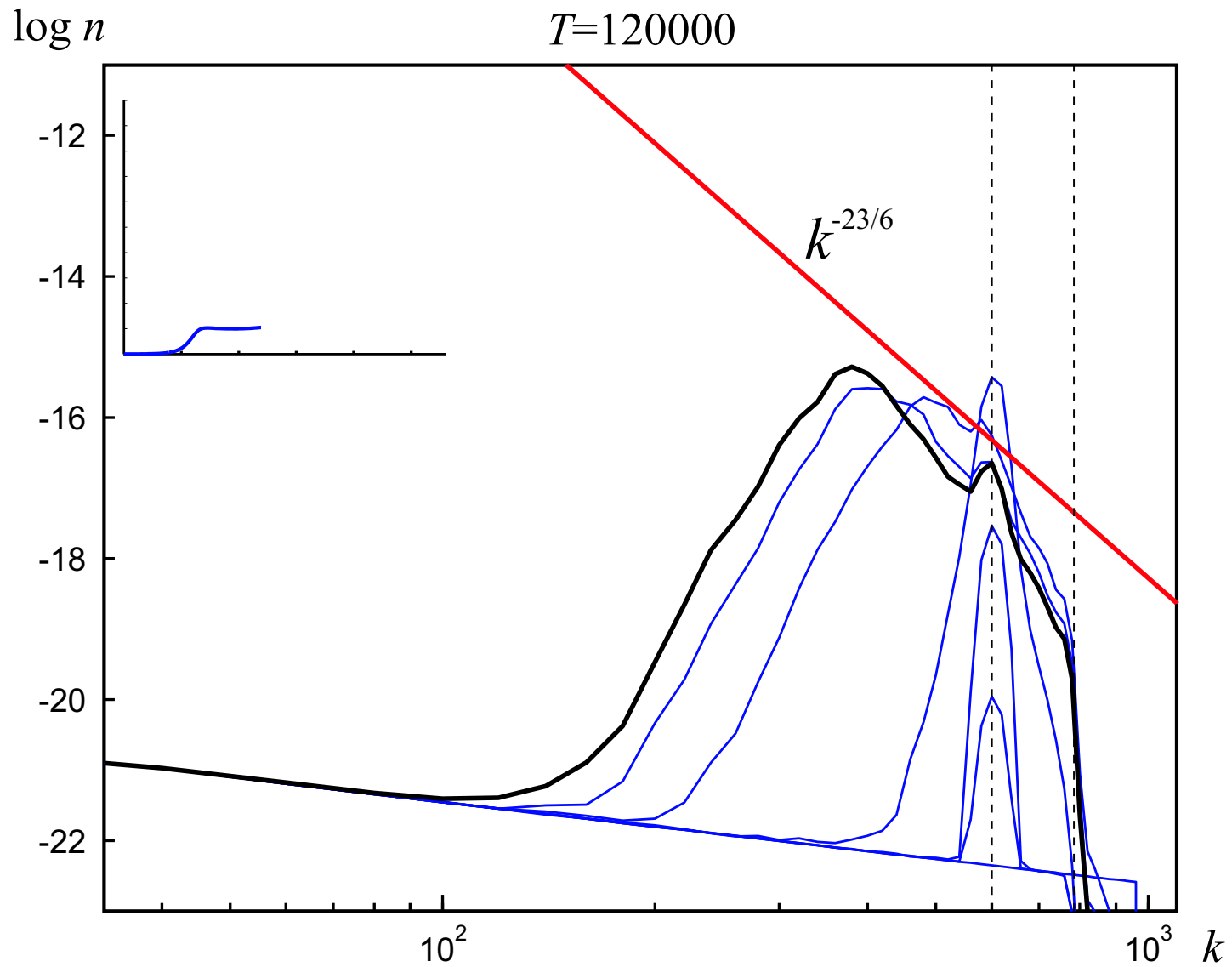
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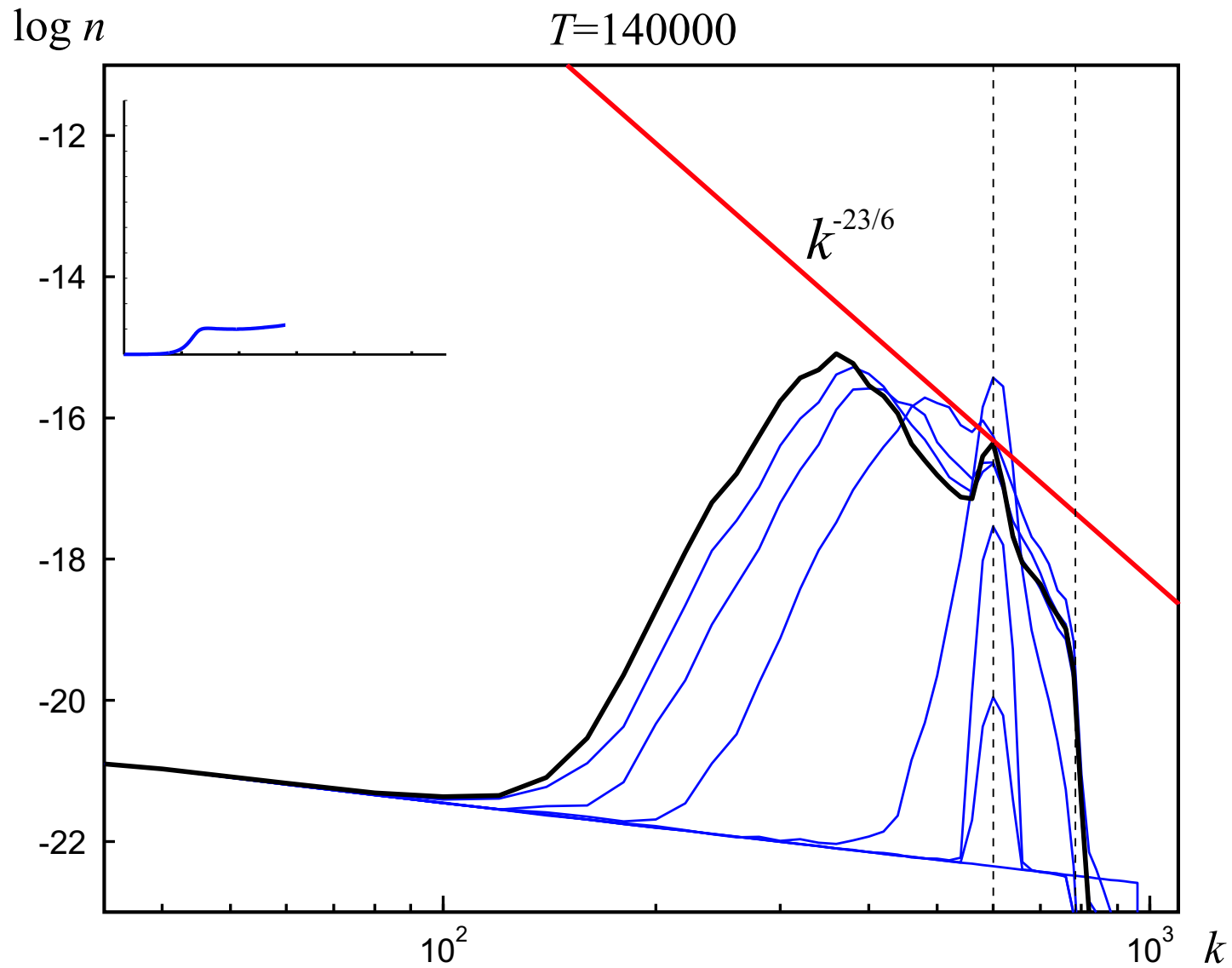
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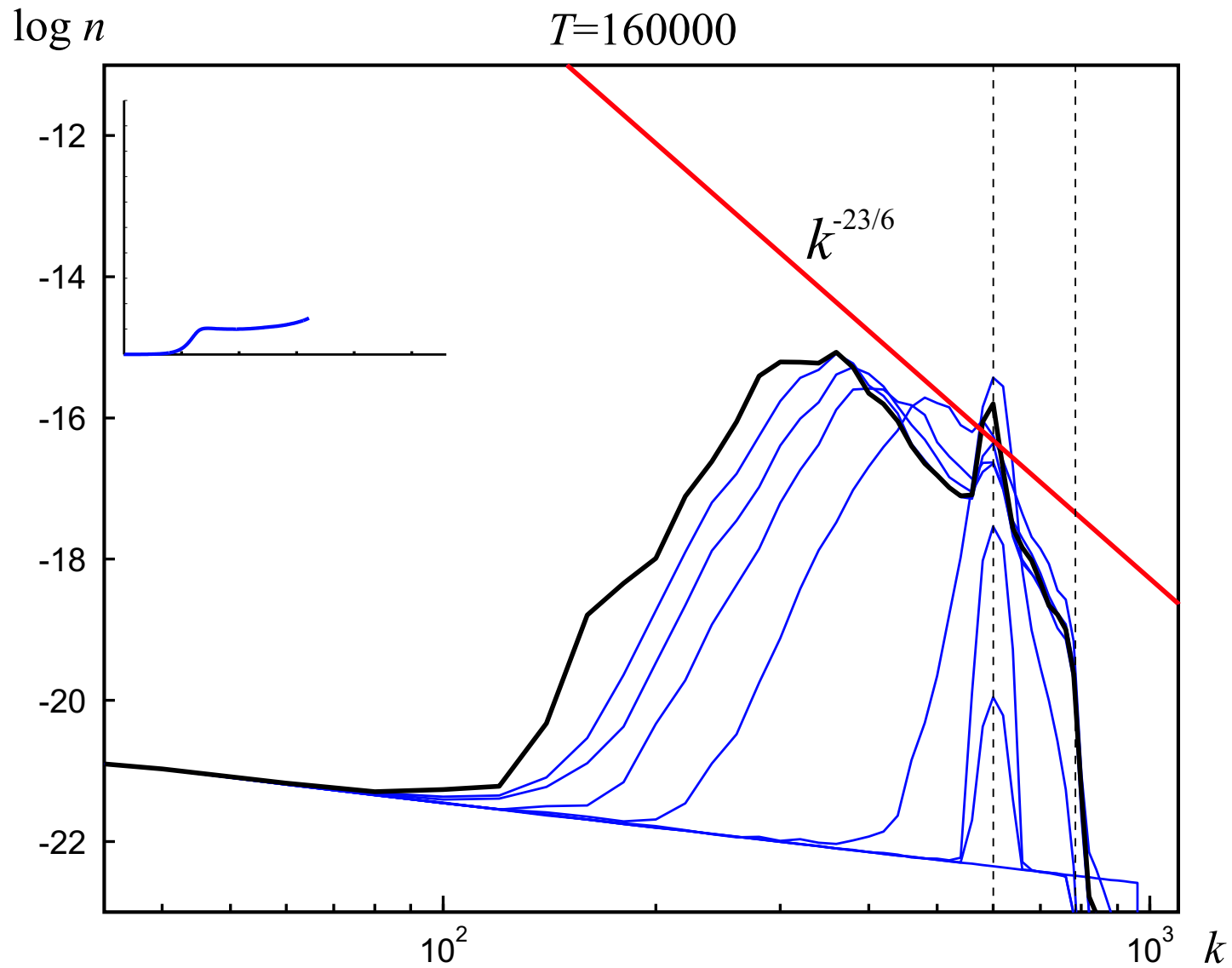
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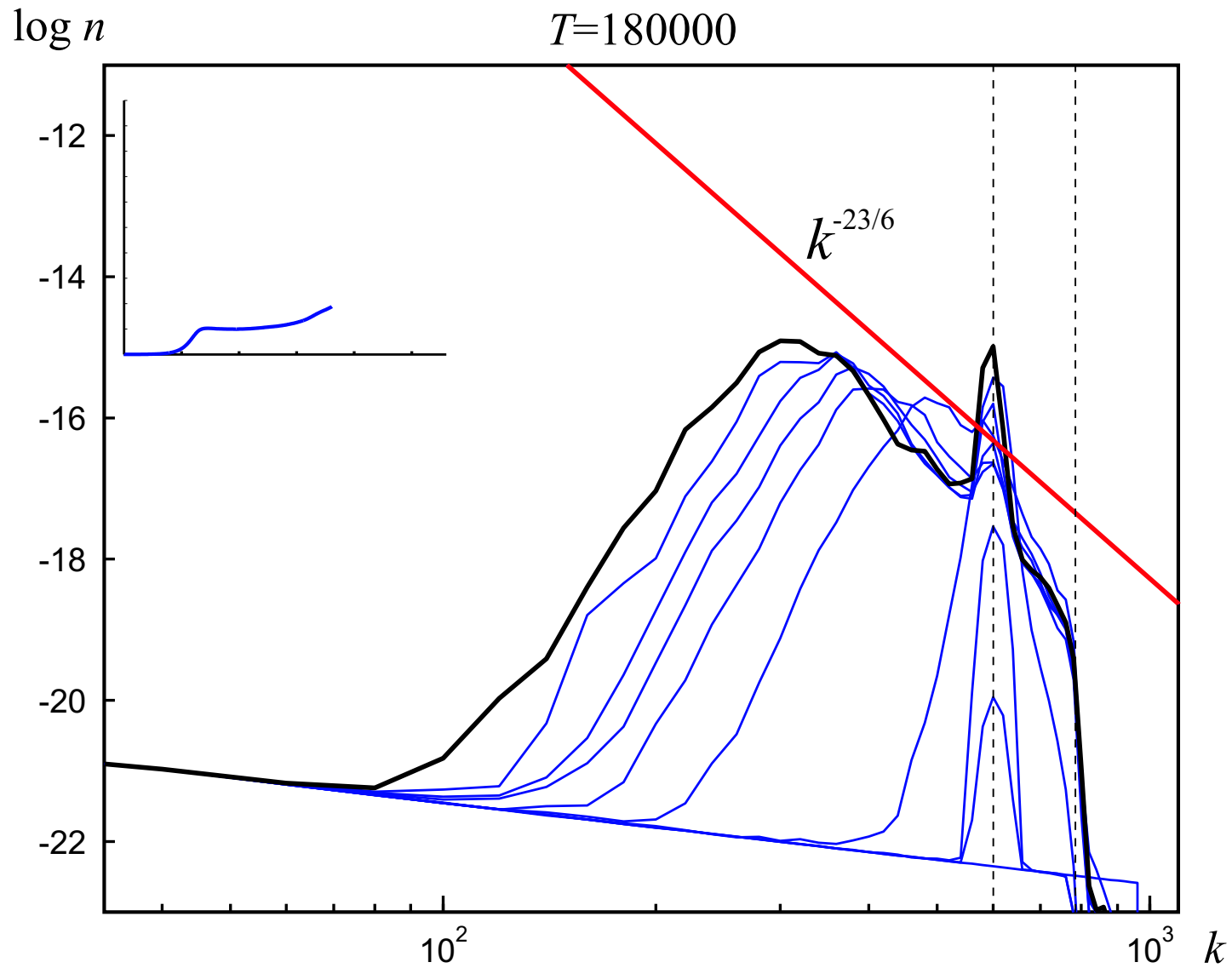
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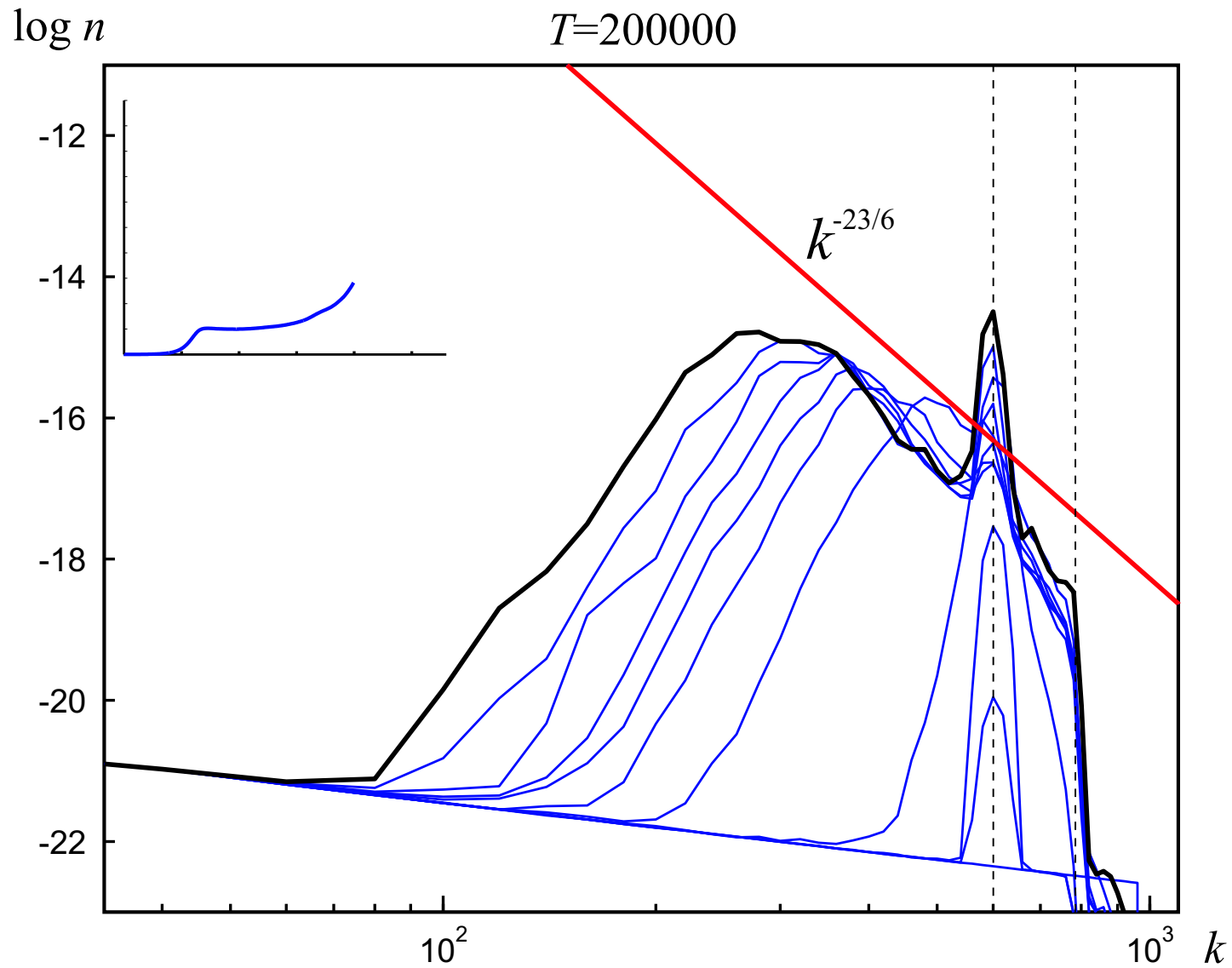
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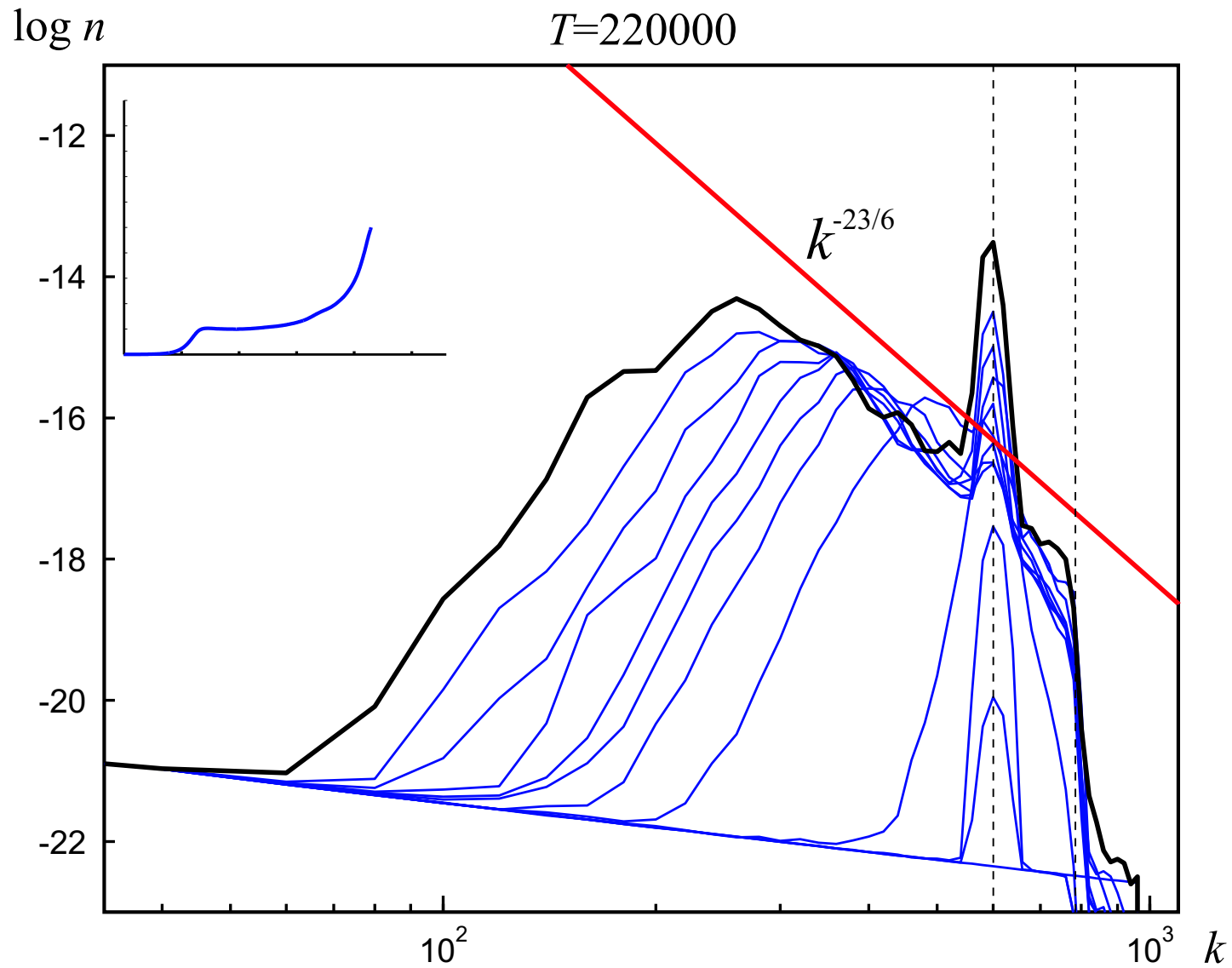
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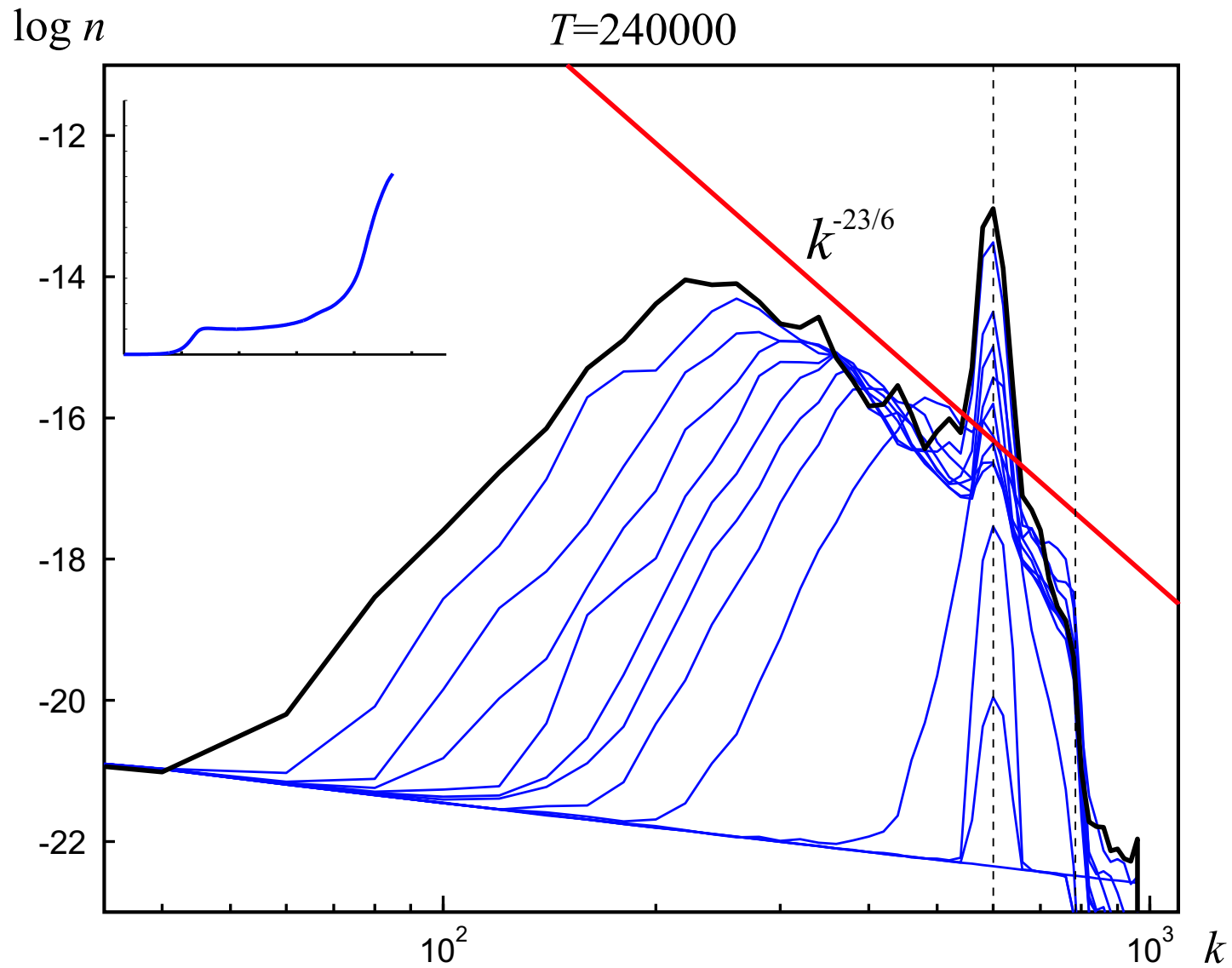
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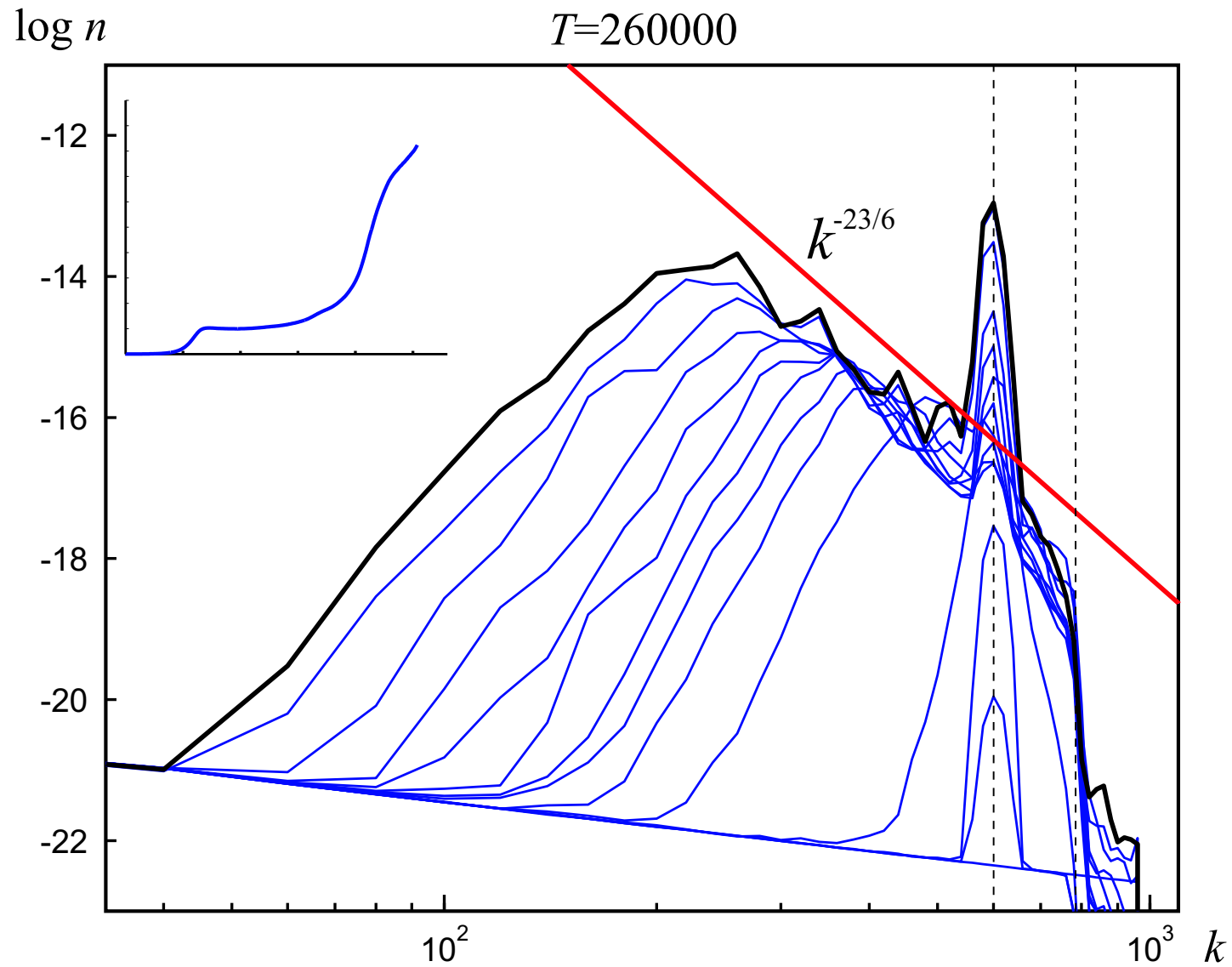
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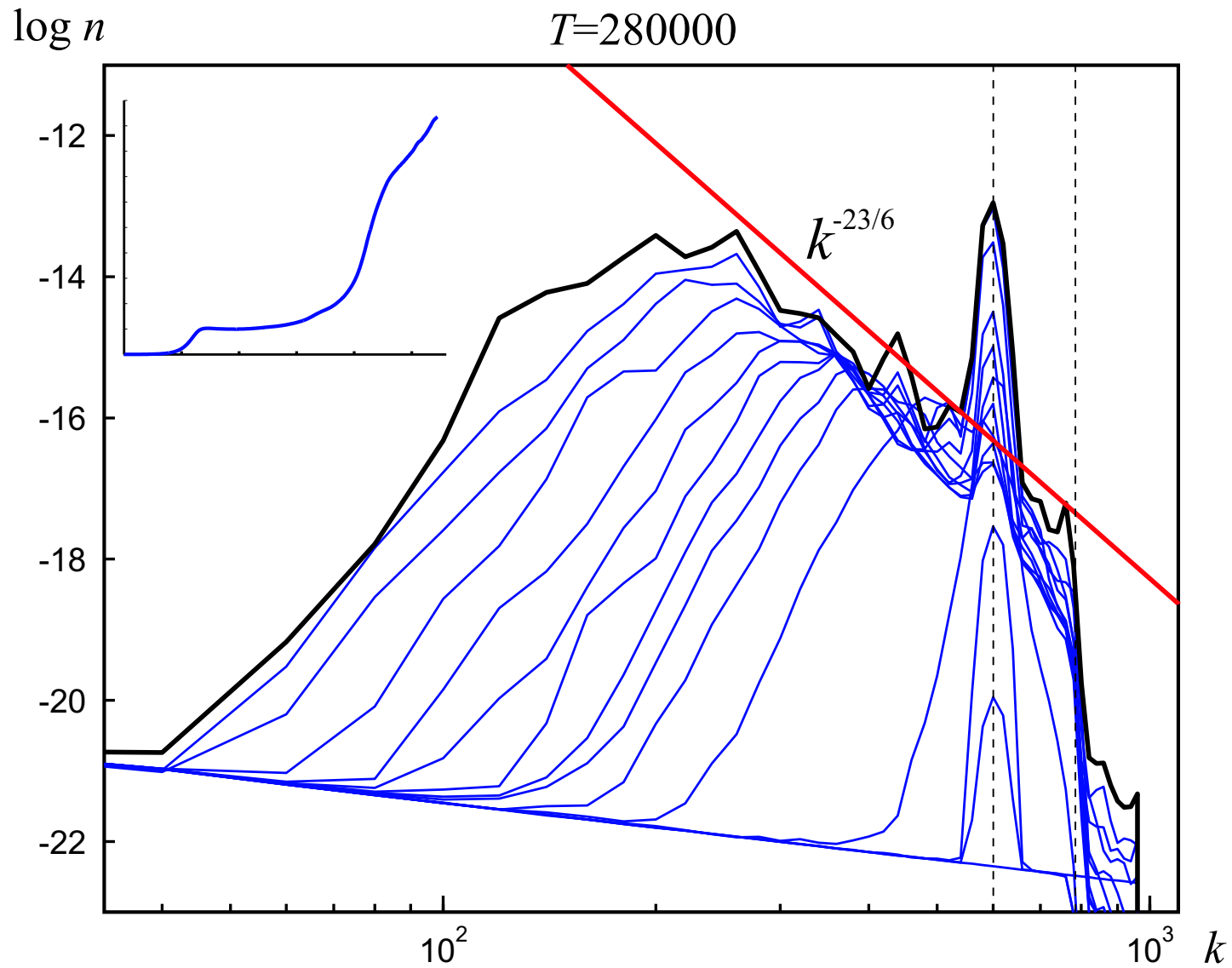
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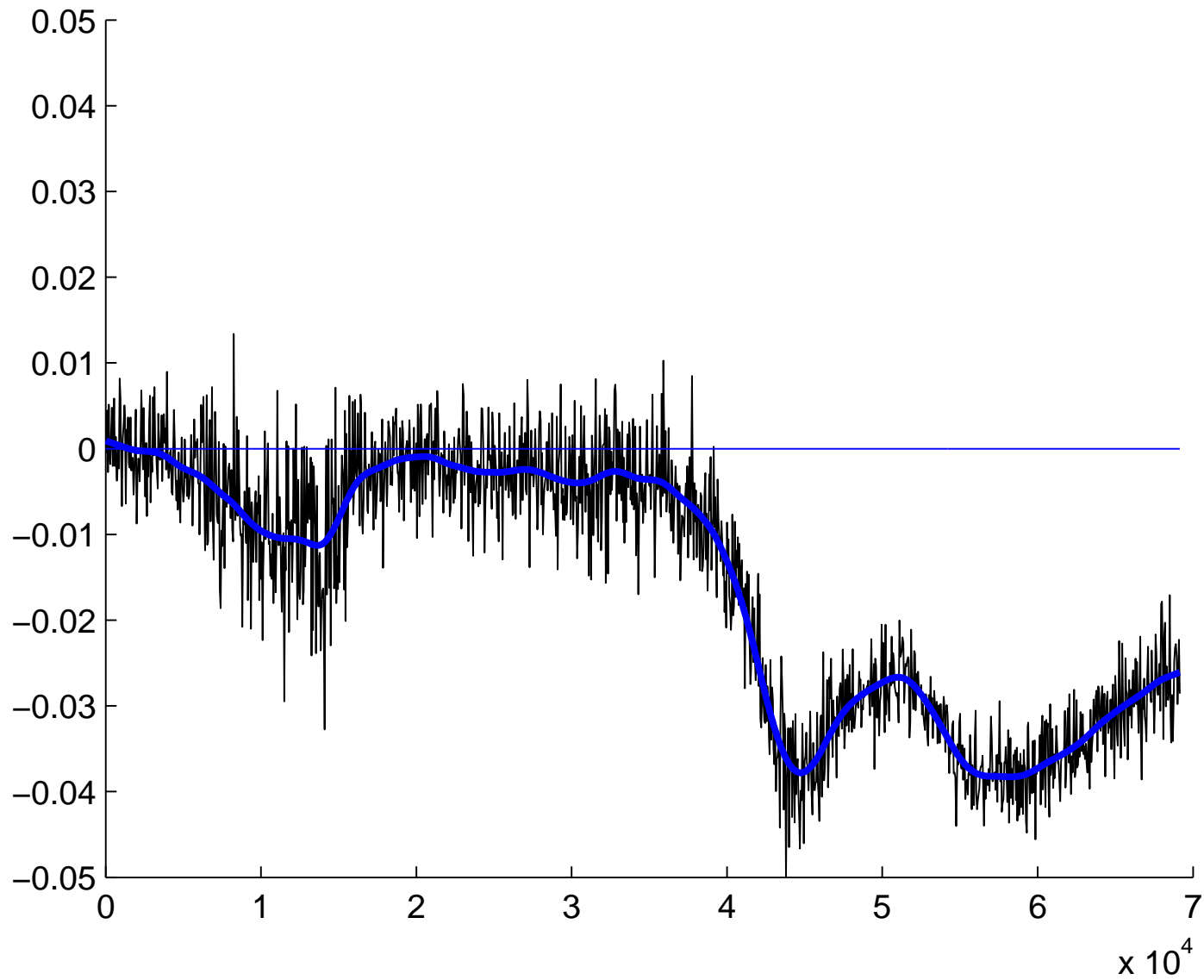
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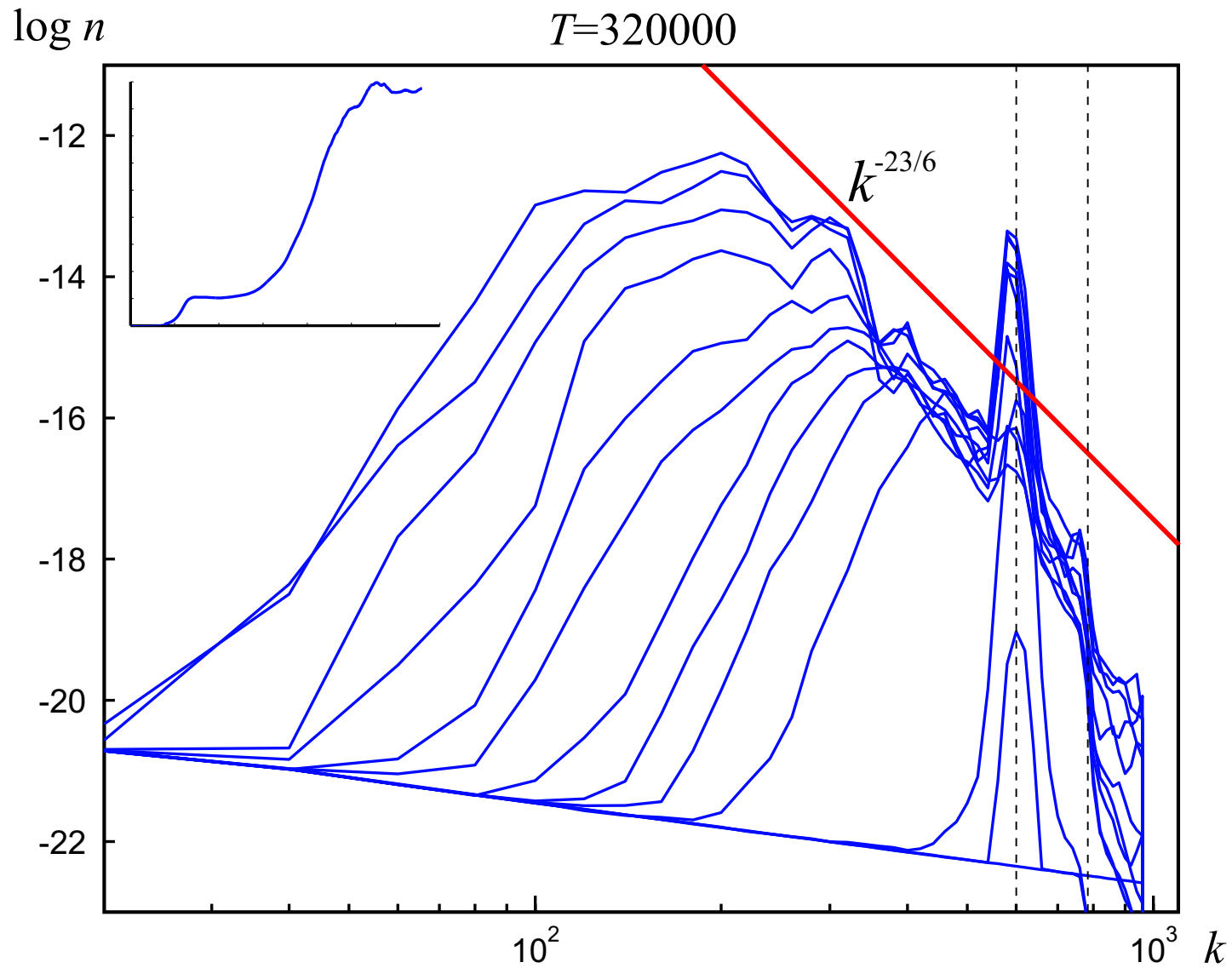
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Forced inverse cascade - kurtosis



Forced inverse cascade - 2nd example



Conclusions

- A method for direct numerical simulation of random water wave fields is proposed.
 - based on the integration of the Zakharov equation
 - uses arbitrary grids
 - includes resonant and near-resonant processes
 - allows computation of higher-order moments
 - allows to include higher-order interactions
 - gives good agreement with the KE when it is applicable