Modeling of swell propagation: direct numerical verification of the theory of weak turbulence

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1 Introduction

Theory of weak turbulence is designed for statistical description of weakly-nonlinear wave ensembles in media with dispersion. The main tool of weak turbulence is kinetic equation for squared wave amplitudes, or a system of such equations. Since the discovery of the kinetic equation for waves by Nordheim (1928) (see also Pierls (1981)) in the context of solid state physics, this quantum-mechanical tool was applied to wide variety of classical problems, including wave turbulence in hydrodynamics, plasmas, liquid helium, nonlinear optics, etc. (see monograph by Zakharov, Falkovich and Lvov (1992)). Such kinetic equations have rich families of exact solutions describing weak – turbulent Kolmogorov spectra. Also, kinetic equations for waves have self-similar solutions describing temporal or spatial evolution of weak – turbulent spectra.

However, the most remarquable example of weak turbulence is wind-driven sea. The kinetic equation describing statistically the gravity waves on the surface of ideal liquid was derived by Hasselmann (1962). Since this time the Hasselmann equation is widely used in physical oceanography as foundation for development of wave-prediction models: *WAM, SWAN* and *WAVEWATCH*: it is quite special between other applications of the theory of weak turbulence due to the strength of industrial impact.

In spite of tremendous popularity of the Hasselmann equation, its validity and applicability for description of real wind-driven sea has never been completely proven. It was criticized by many respected authors, not only in the context of oceanography. There are at least two reasons why the weak–turbulent theory could fail, or at least be incomplete.

The first reason is presence of the coherent structures. The weak-turbulent theory describes only weakly-nonlinear resonant processes. Such processes are spatially extended, they provide weak phase and amplitude correlation on the distances significantly exceeding the wave length. However, nonlinearity also causes another phenomena, much stronger localized in space. Such phenomena – solitons, quasi-solitons and wave collapses are strongly nonlinear and cannot be described by the kinetic equations. Meanwhile, they could compete with weakly-nonlinear resonant processes and make comparable or even dominating contribution in the energy, momentum and wave-action balance. For gravity waves on the fluid surface the most important coherent structures are white – cappings (or wave – breakings), responsible for essential dissipation of wave energy.

The second reason limiting the applicability of the weak-turbulent theory is finite size of any real physical system. The kinetic equations are derived only for infinite media, where the wave vector runs continuous *D*-dimensional Fourier space. Situation is different for the wave systems with boundaries, e.g. boxes with periodical or reflective boundary conditions. The Fourier space of such systems is infinite lattice of discrete eigenmodes. If the spacing of the lattice is not small enough, the whole physics of nonlinear interaction becomes completely different from continuous case.

For these two reasons verification of the weak turbulent theory is an urgent problem, important for the whole physics of nonlinear waves. The verification can be done by direct numerical simulation of the primitive dynamical equations describing wave turbulence in nonlinear medium.

So far all experimentalists tried to check some predictions of the weak-turbulent theory, such as weak-turbulent Kolmogorov spectra. For the gravity wave turbulence the most important is Zakharov-Filonenko spectrum $F_\omega \simeq \omega^{-4}$ (see Zakharov and Filonenko (1966)). At the moment this spectrum was observed in many numerical experiments.

The attempts of verification of weak turbulent theory through numerical simulation of primitive dynamical equations has been started with numerical simulation of 2D optical turbulence by Dyachenko, Newell, Pushkarev and Zakharov (1992), which demonstrated, in particular, co–existence of weak – turbulent and coherent events.

Numerical simulation of 2D turbulence of capillary waves was done by Pushkarev and Zakharov (1996), Pushkarev and

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Zakharov (2000) and Pushkarev (1999). The main result of the simulation is the fact that besides observed classical regime of turbulence with spectrum $F_\omega \simeq \omega^{-4}$ there is non-classical regime of "frozen" turbulence characterized by absence of energy transfer from low to high wave-numbers. The regime of "frozen" turbulence is realized due to absence of exact 3-wave resonances on the discrete grid in Fourier space. Classical regime of Kolmogorov turbulence was found to be realized through excitation of approximate 3-wave resonances on the discrete grid due to nonlinear frequency shifts. Conclusion has been made that in the reality the turbulence of dispersive waves practically always is the mixture of classical Kolmogorov and non-classical "frozen" turbulence, and that the regimes of pure Kolmogorov, or "frozen" turbulence could be achieved as the limiting cases of the excitations level and/or grid sparcity in Fourier space.

Majda, McLaughlin and Tabak (1997) formulated and studied the model of 1D wave turbulence. Later on, this 1D model was intensively studied numerically by two groups of authors (see review article by Zakharov, Dias and Pushkarev (2004)). As the result, corresponding Kolmogorov spectra have been detected. Besides detecting weak-kolmogorov spectra, conclusion has been made, that 1D dispersive wave turbulence is, generally speaking, co-existence of weak-turbulent and coherent collapsing events – quasisolitons.

Numerical simulation of 2D turbulence of surface gravity waves was performed by Onorato, Osborne, Serio, Resio, Pushkarev, Zakharov, Brandini (2002) and Dyachenko, Korotkevich and Zakharov (2003). These simulation demonstrated Zakharov-Filonenko spectrum $F_\omega \simeq \omega^{-4}$ for high-enough resolution of the grid in Fourier space.

Numerical verification of the Hasselmann equation through comparison with dynaical equations was pioneered by Tanaka Tanaka, Yokoyama (2004) and then confirmed by different authors [?].

In current work we directly compare the results produced by numerical solution of Caughy probem for Hasselmann equation with the results of numerical solution of corresponding Caughy problem for dynamical equations. Our resuls show that the evolution of two models differ significantly, if Fourier-space lattice spacing is scarce enough. In the case of refind lattice spacing, we find convergence of the evolution of both systems.

Results of current study seems to represent both significant academic and industrial interests for the following reasons. For the first time the validity of weak turbulence is confirmed from the first principles. It is also clear now, that the results of not only numerical simulation of dynamical equations, but also laboratory experiments in wave tanks, should be used for interpretation of wave phenomena in the open ocean with extra care: observed characteristic wave length should be small enough and level of wave amplitude should be high enough to avoid the effects of "frozen" turbulence in limited system. The same sort of analysis is applicable to wave turbulence experimental set-ups in another context – plasma, solid state and nonlinear optics.

2 Dynamical model

The fluid motion is potential and described by shape of surface $\eta(\vec{r},t)$ and velocity potential $\psi(\vec{r},t)$, evaluated on the surface. These variables satisfy Zakharov (1968) canonical equations:

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \tag{1}$$

Hamiltonian H is presented by the first three terms in expansion on powers of nonlinearity $\nabla\eta$

$$\begin{aligned} &H = H_0 + H_1 + H_2 + ..., \\ &H_0 = \frac{1}{2}\int \left(g\eta^2 + \psi\hat{k}\psi\right) dxdy, \\ &H_1 = \frac{1}{2}\int \eta\left[|\nabla\psi|^2 - (\hat{k}\psi)^2\right] dxdy, \\ &H_2 = \frac{1}{2}\int \eta(\hat{k}\psi)\left[\hat{k}(\eta(\hat{k}\psi)) + \eta\nabla^2\psi\right] dxdy. \end{aligned} \tag{2}$$

Thereafter we put gravity acceleration equal to $g = 1$. Here $\hat{k}$ is a linear integral operator $\hat{k} = \sqrt{-\nabla^2}$ defined in Fourier space trough product of the harmonics $\psi_{\vec{k}} = \frac{1}{2\pi}\int \psi_{\vec{r}} e^{i\vec{k}\vec{r}} dxdy$ and the modulus of the wavenumber $\sqrt{k_x^2 + k_y^2}$. For gravity waves this reduced Hamiltonian describes four-wave interaction.

Dynamical equations (1) take the form

$$\begin{aligned} \dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\ &\quad + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \tfrac{1}{2}\nabla^2[\eta^2\hat{k}\psi] + \tfrac{1}{2}\hat{k}[\eta^2\nabla^2\psi], \\ \dot{\psi} &= -g\eta - \tfrac{1}{2}\left[(\nabla\psi)^2 - (\hat{k}\psi)^2\right] - \\ &\quad - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\nabla^2\psi. \end{aligned} \tag{3}$$

Introduction of the canonical variables $a_{\vec{k}}$

$$a_{\vec{k}} = \sqrt{\frac{\omega_k}{2k}}\eta_{\vec{k}} + i\sqrt{\frac{k}{2\omega_k}}\psi_{\vec{k}}, \tag{4}$$

where $\omega_k = \sqrt{gk}$, transforms equations (1) into

$$\frac{\partial a_{\vec{k}}}{\partial t} = -i\frac{\delta H}{\delta a^*_{\vec{k}}}. \tag{5}$$

Equations (3) were solved numerically in a box $2\pi\times 2\pi$ using the spectral code with periodic boundary conditions with the help of implicit energy-preserving scheme, similar to used in Dyachenko et al. (2003), Dyachenko et al. (2003), Dyachenko et al. (2004). Numerical experiments have been performed on the grids from 256×256 through 512×4096.

3 Weak-turbulent model

Let $\eta(\vec{r},t)$ be a surface elevation, $\psi(\vec{r})$ be a potential on the surface. We assume that density of the fluid $\rho = 1$. The complex amplitude of propagating waves is given by the formula:

$$a_k = \frac{1}{\sqrt{2}}\left[\left(\frac{g}{k}\right)^{1/4}\eta_{\vec{k}} - i\left(\frac{k}{g}\right)^{1/4}\psi_{\vec{k}}\right] \tag{6}$$

In the pair of correlation functions,

The kinetic equation for waves

$$\frac{\partial n_{\vec{k}}}{\partial t} = S_{nl}[n] \tag{7}$$

$$S_{nl}[n] = 2\pi g^2 \int |T_{\vec{k},\vec{k_1},\vec{k_2},\vec{k_3}}|^2 (n_{\vec{k_1}} n_{\vec{k_2}} n_{\vec{k_3}} + n_{\vec{k}} n_{\vec{k_2}} n_{\vec{k_3}}$$
$$- n_{\vec{k}} n_{\vec{k_1}} n_{\vec{k_2}} - n_{\vec{k}} n_{\vec{k_1}} n_{\vec{k_3}}) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})$$
$$\delta(\vec{k} + \vec{k_1} - \vec{k_2} - \vec{k_3})\, d\vec{k_1} d\vec{k_2} d\vec{k_3},$$

where $\omega = \sqrt{gk}$.

Numerical integration of kinetic equation for gravity waves on deep water (Hasselmann equation) was the subject of considerable efforts for last three decades. The “ultimate goal” of the effort – creation of the operational wave model for wave forecast based on direct solution of the Hasselmann equation – happened to be an extremely difficult computational problem due to mathematical complexity of the S_{nl} term, which requires calculation of a three-dimensional integral at every advance in time.

Historically, numerical methods of integration of kinetic equation for gravity waves exist in two “flavors”.

The first one is associated with works of Hasselmann at al. (1985), Dungey and Hui (1985), Masuda (1981), Masuda (1986), Lavrenov (1998) and Polnikov (2001), and is based on transformation of 6-fold into 3-fold integrals using δ-functions. Such transformation leads to appearance of integrable singularities, which creates additional difficulties in calculations of the S_{nl} term.

The second type of models has been developed in works of Webb (1978) and Resio, Tracy (1982) Resio, Perrie (1991) and is currently know as Resio-Tracy model. It uses direct calculation of resonant quadruplet contribution into S_{nl} integral, based on the following property: given two fixed vectors $\vec{k}, \vec{k_1}$, another two $\vec{k_2}, \vec{k_3}$ are uniquely defined by the point “moving” along the resonant curve – locus.

Numerical simulation in the current work was performed with the help of modified version of the second type algorithm. Calculations were made on the grid 71×36 points in the frequency-angle domain $[\omega, \theta]$ with exponential distribution of points in the frequency domain and uniform distribution of points in the angle direction.

Up to now, Resio-Tracy model suffered rigorous testing and is currently used with high degree of trustworthiness:it was tested with respect to motion integrals conservation in the “clean” tests, wave action conservation in wave spectrum down-shift, realization of self – similar solution in “pure swell” and “wind-forced” regimes (see Pushkarev and Zakharov (2000), Pushkarev, Resio and Zakharov (2003), Badulin, Pushkarev, Resio and Zakharov (2005)).

4 Numerical simulation

As initial conditions, we used a Gauss-shaped distribution on a long axis of the wavenumbers plane

$$\begin{cases} |a_{\vec{k}}| = A_i \exp\left(-\frac{1}{2}\frac{\left|\vec{k}-\vec{k}_0\right|^2}{D_i^2}\right), \left|\vec{k}-\vec{k}_0\right| \le 2D_i, \\ |a_{\vec{k}}| = 10^{-12}, \left|\vec{k}-\vec{k}_0\right| > 2D_i, \end{cases} \tag{8}$$
$$A_i = 5\times 10^{-6}, D_i = 30, \vec{k}_0 = (0;150).$$

The initial phases of all the harmonics were random. The average steepness $\mu =< |\nabla\eta| >\simeq 0.115$. To stabilize the computations in the high-frequency region (see Lushnikov and Zakharov (2005)), we introduced artificial damping, mimicking viscosity at small scales, and an artificial smoothing term to the equation for the surface evolution

$$\begin{aligned} &\frac{\partial \psi_{\vec{k}}}{\partial t} \to \frac{\partial \psi_{\vec{k}}}{\partial t} + \gamma_k \psi_{\vec{k}}, \\ &\frac{\partial \eta_{\vec{k}}}{\partial t} \to \frac{\partial \eta_{\vec{k}}}{\partial t} + \gamma_k \eta_{\vec{k}}, \\ &\gamma_k = \begin{cases} 0, k < k_d, \\ -\gamma(k-k_d)^2, k \ge k_d, \end{cases} \\ &k_d = 512, \gamma = 2\times 10^4, \tau = 3.1\times 10^{-4}. \end{aligned} \tag{9}$$

The process of waves evolution can be separated in two steps. On the first stage (about fifty initial wave periods), we observe fast loss of energy and wave action. This effect can be explained by formation of ”slave” harmonics taking their part of motion constants. Initially smooth spectrum becomes very rough. The spectral maximum demonstrates fastdownshift.

With the time step τ, this calculations took about two months on AMD Athlon 64 3500+ computer. During this time, we reached 1500 periods of the wave in the initial spectral maximum.

4.1 Statistics of harmonics

Distribution function of harmonics in damping and damping-free regions Fig.1.

4.2 Artificial viscosity case

Total wave action for artificial viscosity case Fig.2.
Total wave energy for artificial viscosity case Fig.3.
Total wave action for artificial viscosity case Fig.4.
Total wave energy for artificial viscosity case Fig.5.
Total wave action for artificial viscosity case Fig.6.
Total wave energy for artificial viscosity case Fig.7.
Total wave action for artificial viscosity case Fig.8.
Total wave energy for artificial viscosity case Fig.9.
Angle-averaged spectrum as a function of time Fig.10.

4.3 No viscosity case

Total wave action for artificial viscosity case Fig.2.
Total wave energy for artificial viscosity case Fig.3.

Total wave action for artificial viscosity case Fig.4.
Total wave energy for artificial viscosity case Fig.5.
Total wave action for artificial viscosity case Fig.6.
Total wave energy for artificial viscosity case Fig.7.
Total wave action for artificial viscosity case Fig.8.
Total wave energy for artificial viscosity case Fig.9.
Angle-averaged spectrum as a function of time Fig.10.

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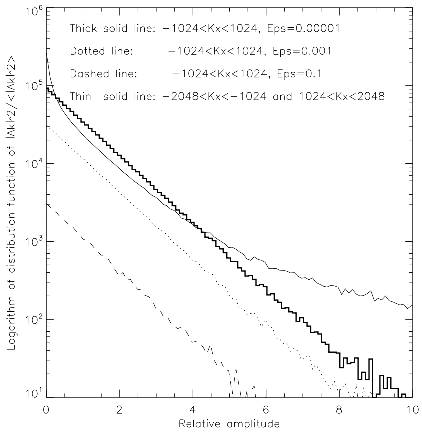

Fig. 1. Mean wave frequency as a function of time for artificial viscosity case.

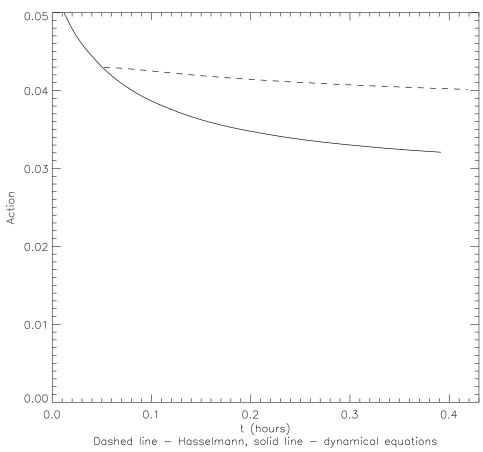

Fig. 2. Total wave action as a function of time for artificial viscosity case.

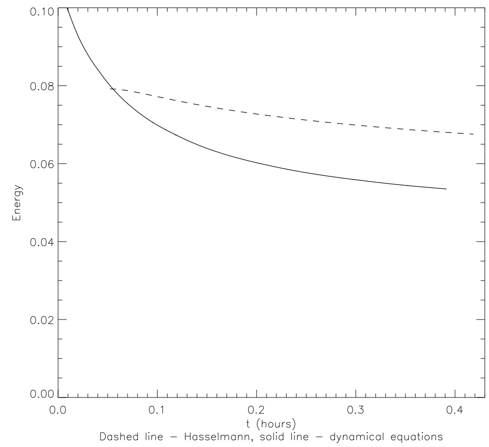

Fig. 3. Total wave energy as a function of time for artificial viscosity case.

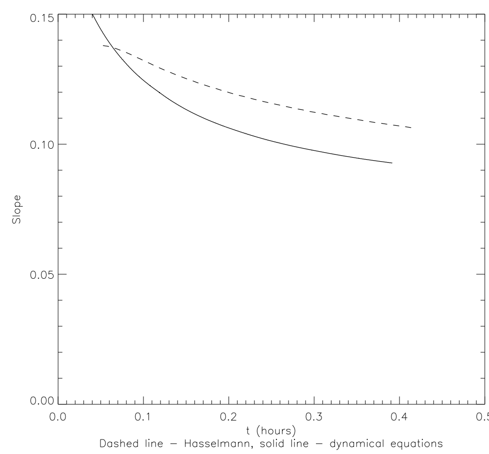

Fig. 4. Meane wave slope as a function of time for artificial viscosity case.

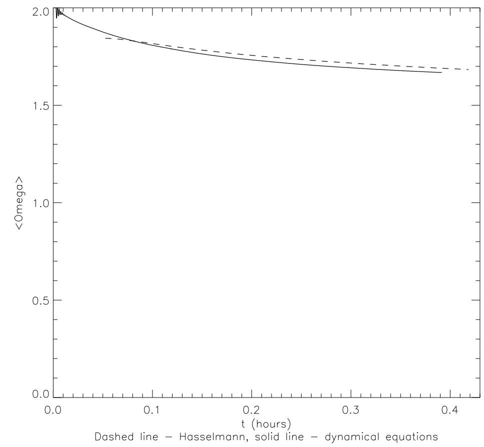

Fig. 5. Mean wave frequency as a function of time for artifi cial viscosity case.

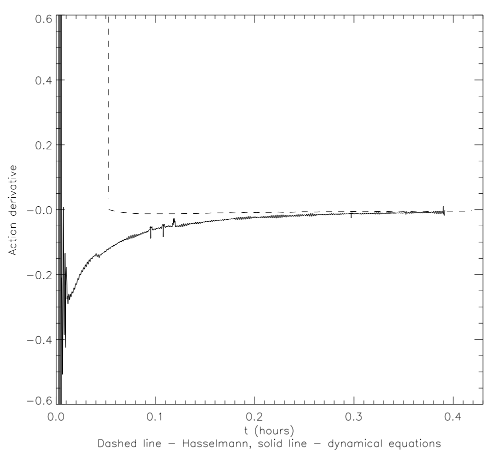

Fig. 6. Total wave action as a function of time for artifi cial viscosity case.

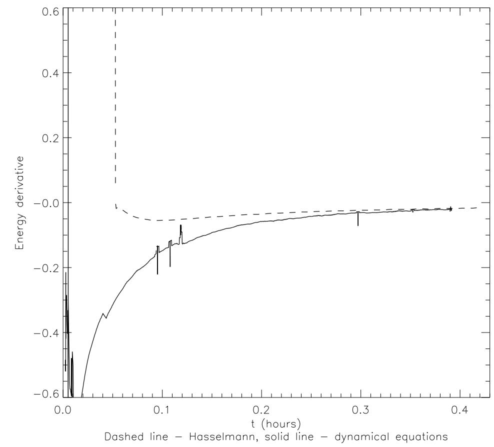

Fig. 7. Total wave energy as a function of time for artifi cial viscosity case.

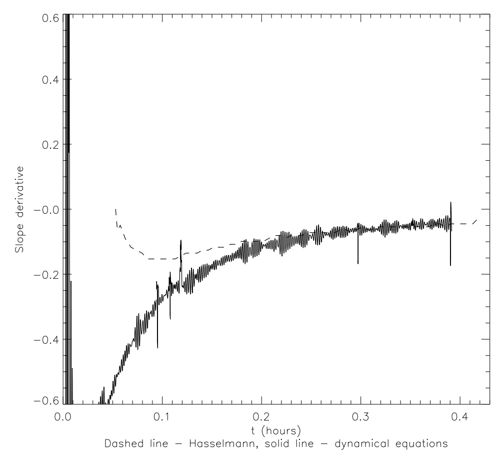

Fig. 8. Meane wave slope as a function of time for artifi cial viscosity case.

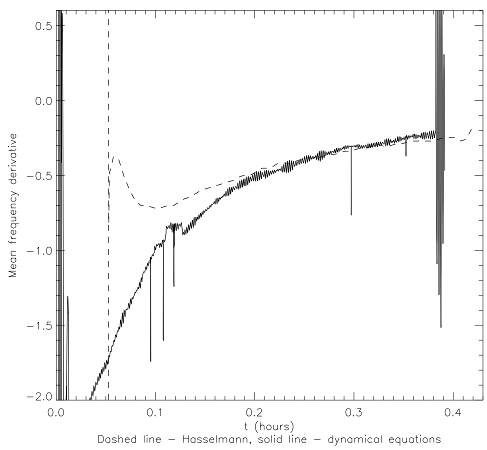

Fig. 9. Mean wave frequency as a function of time for artificial viscosity case.

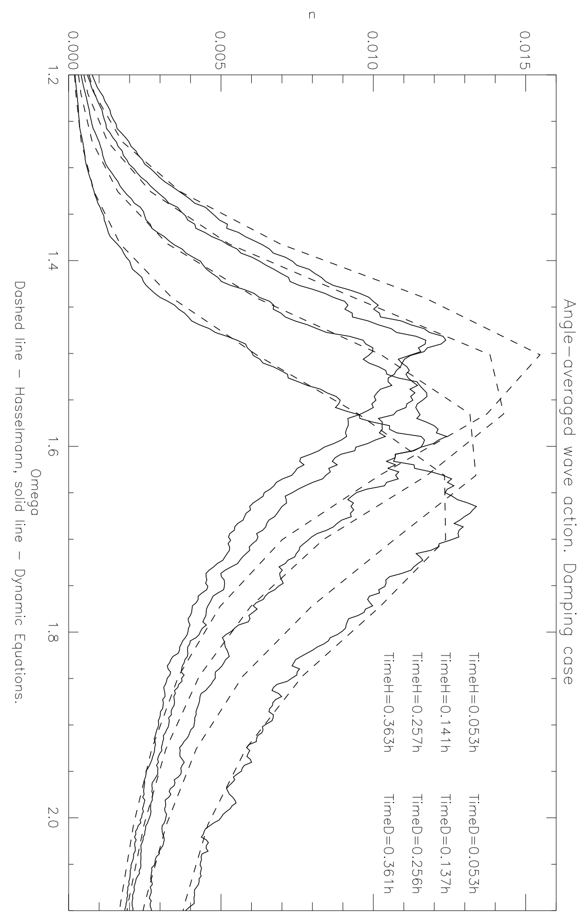

Fig. 10. Mean wave frequency as a function of time for artificial viscosity case.