Freak waves. Theory and experiments

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Abstract

We present numerical simulation of several problem related to free surface hydrodynamics: a) when nonlinear Shredinger approximation breaks, b) nonlinear stage of modulation instability - freak wave formation, c) stability analysis of the free surface hydrodynamics. Simulation is done using two-dimensional code, based on conformal mapping of the fluid to the lower half-plane.

Key words: free surface, freak wave, Nonlinear Shredinger equation, modulation instability
PACS: 02.60.Cb, 47.15.Hg, 92.10.Dh, 92.10Hm

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1 Introduction

Waves of anomalously large size, alternatively called freak, rogue or giant waves are a well-documented hazards for mariners (see, for instance Smith (1976), Dean (1990), Chase (2003)). These waves are responsible for loss of many ships and many human lives. Freak waves could appear in any place of the world ocean (see Earle (1975), Mori et al. (2002), Divinski et al. (2004)); however, in some regions they are more probable than in the others. One of the regions where freak waves are especially frequent is the Agulhas current of the South-East coast of South Africa (see Gerber (1996), Gutshabash et al. (1986), Irvine and Tilley (1988), Lavrenov (1998), Mallory (1974)). In the paper by Peregrine (1976) it was suggested that in areas of strong current such as the Agulhas, giant waves could be produced when wave action

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Preprint submitted to Elsevier Science *6 December 2005*

is concentrated by reflection into a caustic region. According to this theory, a variable current acts analogously to an optic lens to focus wave action. The caustic theory of freak waves was supported since that time by works of many authors. Among them Smith (1976), Gutshabash et al. (1986), Irvine and Tilley (1988), Sand et al. (1990), Gerber (1987), Gerber (1993), Slunyaev et al. (2002), Kharif and Pelinovsky (2003). The statistics of caustics with application to calculation of the freak wave formation probability was studied in the paper of White and Fornberg (1998).

On our opinion, a connection between freak wave generation and caustics for swell or wind-driven sea is the indisputable fact. However, this is not the end of the story. Focusing of ocean waves by an inhomogeneous current is a pure linear effect. Meanwhile, no doubts that freak waves are essentially nonlinear objects. They are very steep. In the last stage of their evolution, the steepness becomes infinite, forming a "wall of water". Before this moment, the steepness is higher than one for the limiting Stokes wave. Moreover, a typical freak wave is a single event (see, for instance Divinski et al. (2004). Before breaking it has a crest, three-four (or even more) times higher than the crests of neighbor waves. The freak wave is preceded by a deep trough or "hole in the sea". A characteristic life time of a freak wave is short - ten of wave periods or so. If the wave period is fifteen seconds, this is just few minutes. Freak wave appears almost instantly from a relatively calm sea. Sure, these peculiar features of freak waves cannot be explained by a linear theory. Focusing of ocean waves creates only preconditions for formation of freak waves, which is a strongly nonlinear effect.

It is natural to associate appearance of freak waves with the modulation instability of Stokes waves. This instability is usually called after Benjamin and Feir, however, it was first discovered by Lighthill (1965). The theory of instability was developed independently by Benjamin and Feir (1967) and by Zakharov (1966). Feir (1967) was the first who observed the instability experimentally in 1967.

Slowly modulated weakly nonlinear Stokes wave is described by nonlinear Shrödinger equation ($NLSE$), derived by Zakharov (1968). This equation is integrable (see Zakharov and Shabat (1972)) and is just the first term in the hierarchy of envelope equations describing packets of surface gravity waves. The second term in this hierarchy was calculated by Dysthe (1979), the next one was found a few years ago by Trulsen and Dysthe (1996).

One cannot deny some advantages achieved by the use of the envelope equations. Results of many authors agree in one important point: nonlinear development of modulation instability leads to concentration of wave energy in a small spatial region. This is a "hint" regarding possible formation of freak wave. On the other hand, it is clear that the freak wave phenomenon cannot be explained in terms of envelope equations. Indeed, $NLSE$ and its generalizations are derived by expansion in series on powers of parameter $\lambda \simeq \frac{1}{Lk}$, where k is a wave number, L is a length of modulation. For real freak wave $\lambda \sim 1$ and any "slow modulation expansion" fails. However, the analysis in the framework of the NLS-type equations gives some valuable information about formation of freak waves.

The most direct way to prove how far envelope equations can be used to describe freak wave formation is a direct numerical solution of Euler equation, describing potential oscillations of ideal fluid with a free surface in a gravitational field. This solution can be made by the method published in several articles (Dyachenko et al. (1996a), Zakharov et al. (2002), Zakharov (1998)). This method is applicable in $1+1$ geometry; it includes conformal mapping of fluid bounded by the surface to the lower half-plane together with "optimal" choice of variables, which guarantees well-posedness of the equations (Dyachenko (2005)). Earlier, in the paper Dyachenko and Zakharov (2005) we studied the nonlinear stage of modulation instability for Stokes waves of steepness $\mu = ka \simeq 0.15$.

In the present article we perform similar experiment for waves of different steepness. We start with the Stokes waver train, perturbed by a long wave. We observe development of modulation instability and finally, the explosive formation of the freak wave that is pretty similar to waves observed in natural experiments.

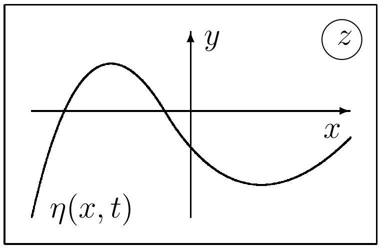
Also we study numerically the limits of applicability of the envelope equations. In particular, we interested in the question how does $NLSE$ fails, and is it possible to improve $NLSE$ keeping the envelope approximation?

2 Conformal equations for 2D fluid

Consider potential irrotational flow with potential $\phi(x, y, t)$

$$\triangle\phi(x, y, t) = 0$$

in the two dimensional domain below the fluid surface



with boundary conditions at the surface:

$$\left[\begin{array}{l} \frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=\frac{P}{\rho}, \\ \frac{\partial\eta}{\partial t}+\eta_x\phi_x=\phi_y \end{array}\right| \text{ at } y=\eta(x,t). \qquad (2.1)$$

and at the infinity:

$$\frac{\partial\phi}{\partial y}=0, y\to-\infty,$$
$$\frac{\partial\phi}{\partial x}=0, |x|\to\infty.$$

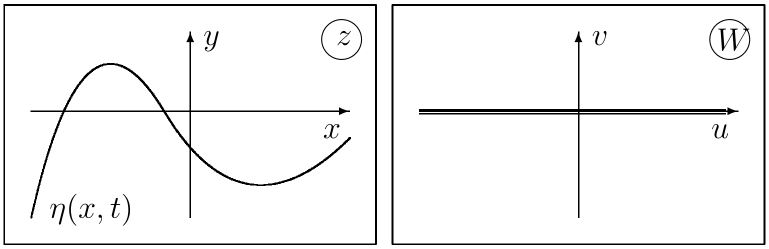
Following Dyachenko et al. (1996a) and Dyachenko (2001) let us perform conformal mapping of the domain on the plane $z=x+iy$,

$$-\infty<x<\infty, \quad -\infty<y\le\eta(x,t)\,,$$

to the lower half-plane

$$-\infty<u<\infty, \quad -\infty<v\le 0\,,$$

on the plane $W=u+iv$.



After this mapping, the surface profile is given parametrically by

$$y=y(u,t), \quad x=u+\tilde{x}(u,t)\,.$$

Functions y and $\tilde{x}$ are coupled by the relations:

$$y=\hat{H}\tilde{x} \qquad\qquad \tilde{x}=-\hat{H}y\,.$$

Here $\hat{H}$ is the Hilbert transformation,

$$\hat{H}f(u)=\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f(u')}{(u'-u)}du'\,.$$

For Fourier harmonics $y_k=i\mathrm{sign}(k)x_k$.

If *conformal mapping* has been applied then it is naturally introduce complex analytic functions

$$Z = x + iy, \qquad \text{and complex velocity potential} \quad \Phi = \Psi + i\hat{H}\Psi.$$

$$\begin{aligned} Z_t &= iUZ_u, \\ \Phi_t &= iU\Phi_u - \hat{P}(\frac{|\Phi_u|^2}{|Z_u|^2}) + ig(Z - u). \end{aligned}$$

U is a complex transport velocity:

$$U = \hat{P}(\frac{-\hat{H}\Psi_u}{|Z_u|^2}). \qquad\qquad u \rightarrow w$$

Projector operator $\hat{P}(f) = \frac{1}{2}(1 + i\hat{H})(f)$.

It turned out, that the equations can be simplified just by changing variables. Introduce instead of $Z(w,t)$ and $\Phi(w,t)$ another analytic functions $R(w,t)$ and $V(w,t)$

$$R = \frac{1}{Z_w}, \qquad\qquad \Phi_w = -iVZ_w.$$

$$\begin{aligned} R_t &= i\left[UR' - U'R\right], \\ V_t &= i\left[UV' - R\hat{P}(V\bar{V})'\right] + g(R - 1). \end{aligned} \tag{2.2}$$

Complex transport velocity U is defined via $\hat{P}$ in a following way

$$U = \hat{P}(V\bar{R} + \bar{V}R).$$

All numerical simulation in this article is done using these equations (2.2). It should be mentioned here, that recently Ruban (2005) suggested analytical model "improving" these equations, which includes weak three-dimensional effects. 3D there is a small corrections to the exact equations (2.2).

3 Stability of the equations

In this section we discuss numerical stability of the solution of (2.2 with respect to small scale perturbations. In particular the stability of the Stokes wave is considered. No doubts we have in mind stability of the round-off perturbations.

We applied spectral code to calculate space derivatives and projector operator. For time integration we used standart Runge-Kutta method of the fourth order. No filtering were used.

Equation for stationary progressive wave is well-known, and for conformal variables it can be found, for example, in Dyachenko et al. (1996b):

$$y = \frac{c^2}{2g}(1 - \frac{1}{|Z_u|^2}),$$

(We look for the solution $y(u - ct)$.

Analytical solution for it is unknown, and one must apply some numerical procedure to get the solution. This equation can be rewritten in the form, much more convenient for numerics:

$$Z_u = 1 + \frac{2g}{c^2}\hat{P}(yZ_u), \tag{3.3}$$

and we can use simple iteration procedure to obtain solution along with the velocity of the wave c:

$$Z_u^{(n+1)} = 1 + \frac{2g}{{c^{(n)}}^2}\hat{P}(y^{(n)}Z_u^{(n)}),$$

Thus we use the solution of (3.3) in the periodic domain 2π with full wave height (from trough to crest) H so, that

$$H \simeq 0.6975.$$

Fourie spectrum of this solution along with perturbation in the harmonic with $k = 200$ is shown in Figure 1. Amplitude of the harmonic with $k = 200$ is 10^{-10}.

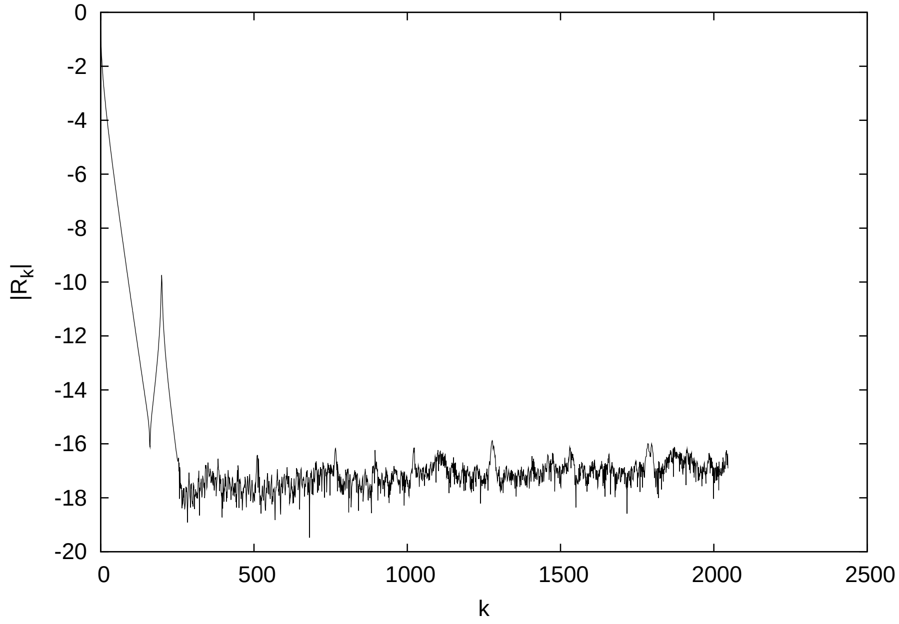


Figure 1. Initial perturbation to the Stokes wave.

After $\simeq 70$ wave periods one can see in Figure 2 that level of perturbation does not increase, and (what is much more important!) there is no instability in high k.

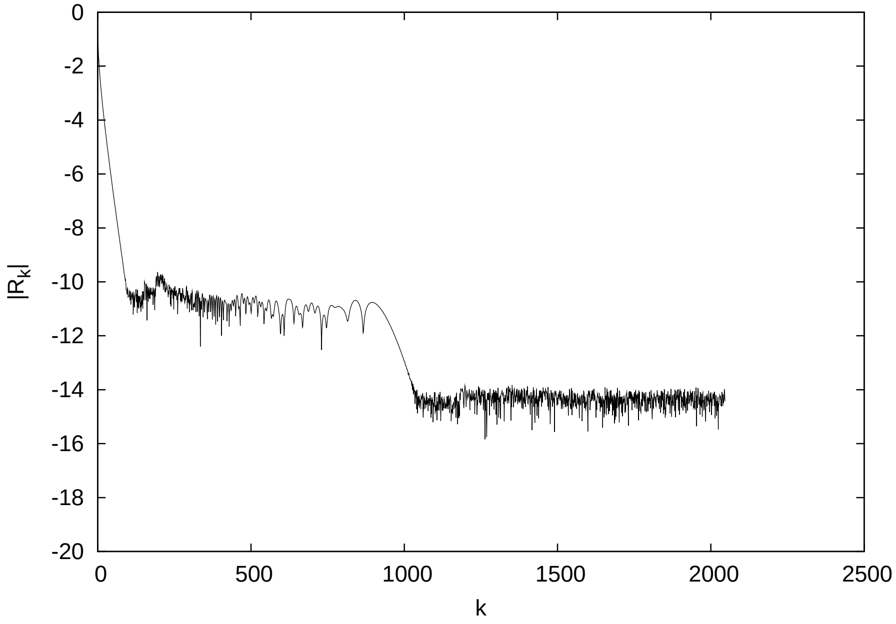


Figure 2. Perturbation and Stokes wave after T=460.

To compare we present here the results of numerical simulation of the same initial value problem, but formulated for different variables. Instead of R one can use Z, comnformal map itself. For this variables equations (2.2) transform to the following form:

$$\begin{aligned} Z_t &= iUZ', \\ V_t &= i\left[UV' - \frac{1}{Z'}\hat{P}(V\bar{V})'\right] + g(\frac{1}{Z'} - 1). \end{aligned} \tag{3.4}$$

and complex transport velocity U now is

$$U = \hat{P}(\frac{V}{\bar{Z}'} + \frac{\bar{V}}{Z'}).$$

Initial condition with perturbation are show in Figure 3.

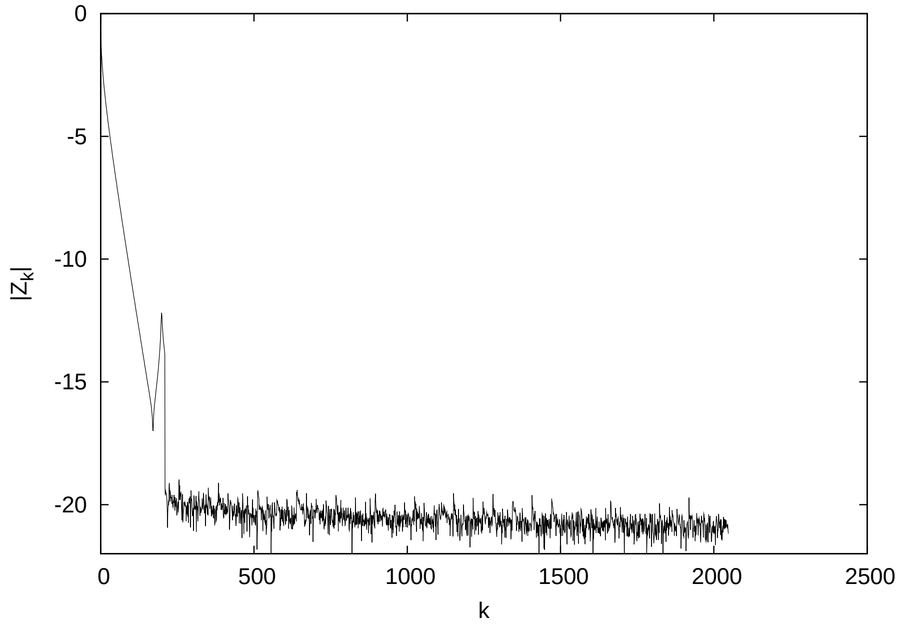


Figure 3. Initial perturbation to the Stokes wave for $V-Z$ variables.

After $\simeq 80$ wave periods one can see in Figure 4 that level of perturbation does not increase, but there is some weak instability in high k.

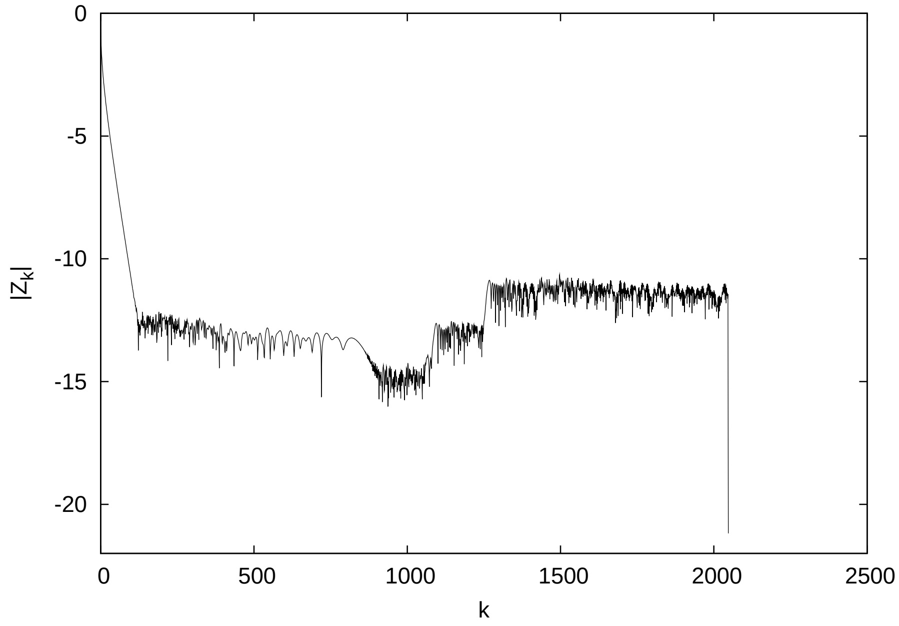


Figure 4. Perturbation and Stokes wave after T=530 for $V-Z$ variables.

Right choice of variables is very important for numerical simulations. As it was shown in Dyachenko (2005), equations (2.2) pass through "frozen coefficients" analysis.

4 Freak wave formation

Here we study modulation instability of uniform wave train of Stokes wave. Question of great interest is the nonlinear stage of modulation instability. Here and everywhere below we do simulation in piriodic domain $L = 2\pi$ and

$$g = 1.$$

- The shape of Stokes progressive wave is given by:

  $$y = \frac{c^2}{2g}(1 - \frac{1}{|Z_u|^2}),$$

  while Φ is related to the surface as

  $$\Phi = -c(Z - u), \qquad V = ic(R - 1).$$

  The amplitude of the wave $\frac{h}{L}$ is the parameter for initial condition. (For the sharp peaked limiting wave $\frac{h}{L} \simeq 0.141$)
- Put 100 such waves with small perturbation in the periodic domain of 2π.

In such a way we prepared initial wave train with the steepness $\mu \simeq 0.095$ Main Fourier harmonic of this wave train is $k = 100$. Similar problem was studied in Song and Banner (2002). But instead of long wavetrain they studied evolution of small group of waves.

For perturbaion small value for Fourier harmonic with $k_p = 1$ was set. So, that

$$R_k = R_k^{\text{unperturbed}} + 0.05 R_{100} \exp^{-ik_p u}.$$

Surface profile of this initial condition is shown in Figure 5

Fourie spectrum of the initial condition is shown in Figure 6 and Figure 7.

After sufficient large time, which is more than 1300 wave periods one can observe freak wave formation, as it is shown in Figure 8. Freak wave grows from mean level of waves to its maximal value for several wave periods, than vanishes or breaks.

Detailed view at the freak wave at the moment of maximal amplitude is shown in Figure 9.

During numerical simulation of the final stage of freak wave formation, resolution must be increased to resolve high curvature of the surface profile. To do this we have ben increasing number of Fourier harmonics, which reached 2^{20} at the end ($T = 802.07$). Fourier coefficients of R_k are shown in Figure 10.

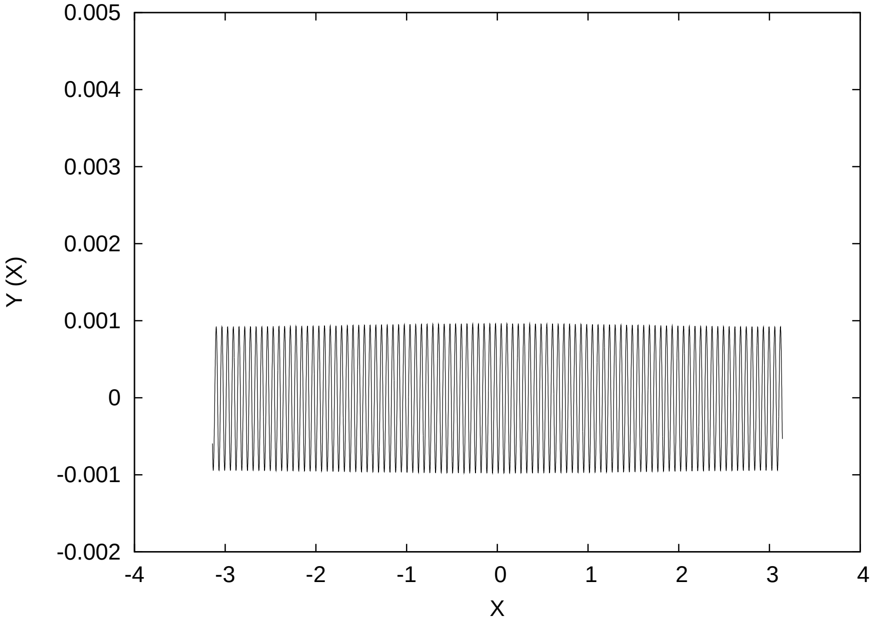


Figure 5. Initial profile of the wave train

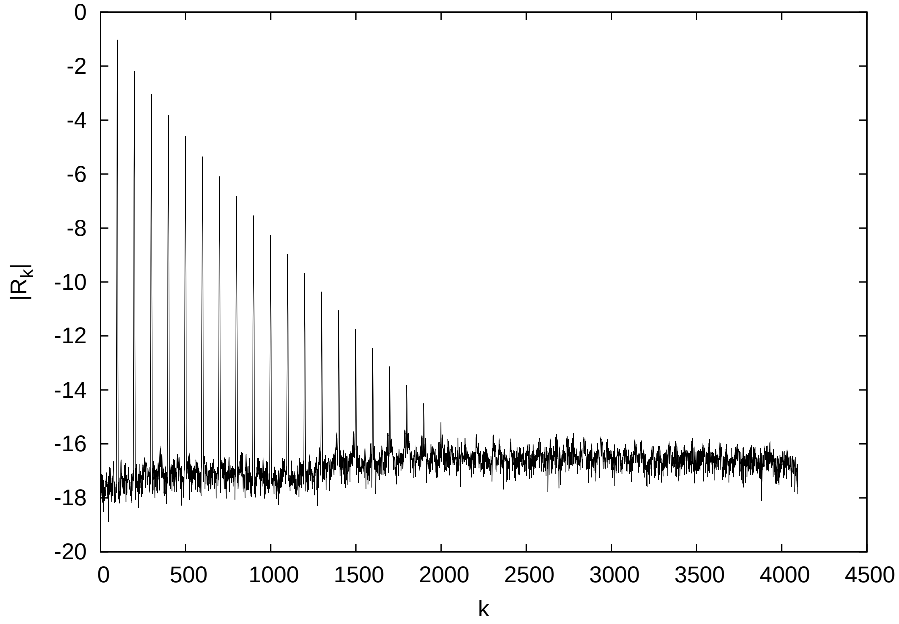


Figure 6. Fourier coefficients $|R_k|$ for initial condition ($\mu \simeq 0.095$).

If amplitude of the wave train is large, than freak wave may eventually break. Such a picture is presented in the Figure 11, which is correspons to the other numerical simulation with the initial steepness $\mu \simeq 0.14$.

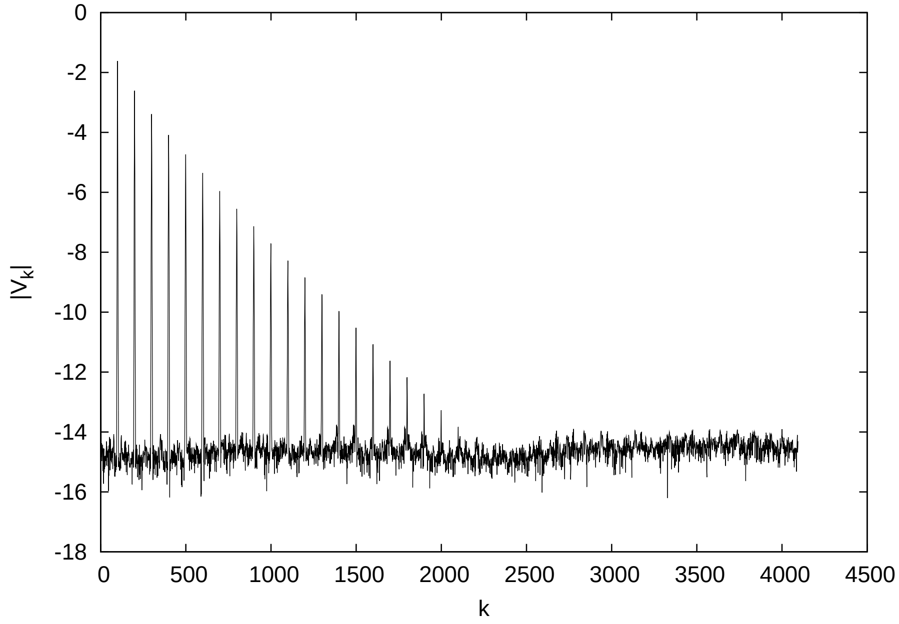


Figure 7. Fourier coefficients $|V_k|$ for initial condition ($\mu \simeq 0.095$).

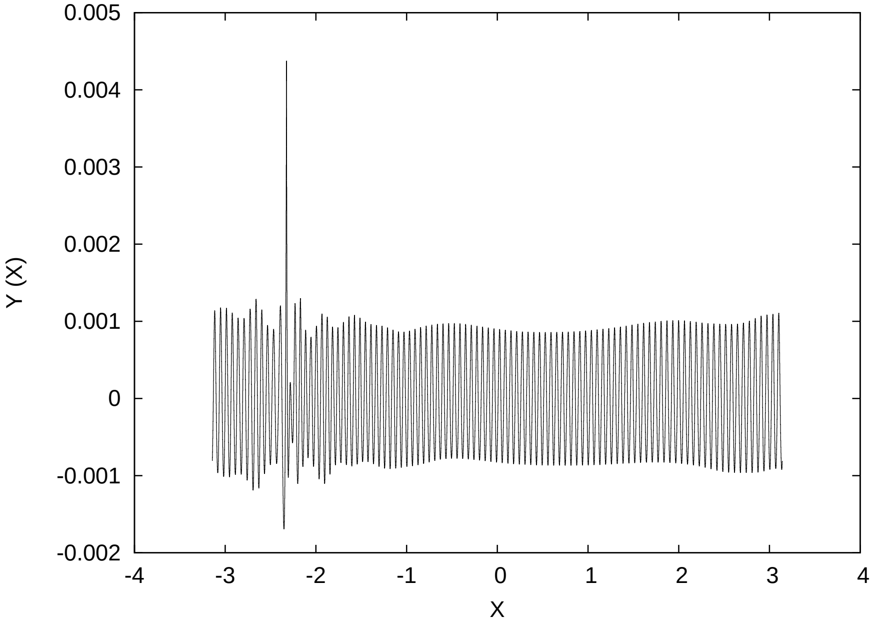


Figure 8. Freak wave on the surface profile. $T = 802.07$

5 Exact equations and nonlinear Shredinger approximation

Evolution of weakly nonlinear Stokes wavetrain can be described by nonlinear Shrödinger equation ($NLSE$), derived by Zakharov (1968). This equation is integrable (see Zakharov and Shabat (1972)) and is just the first term in the hierarchy of envelope equations describing packets of surface gravity waves.

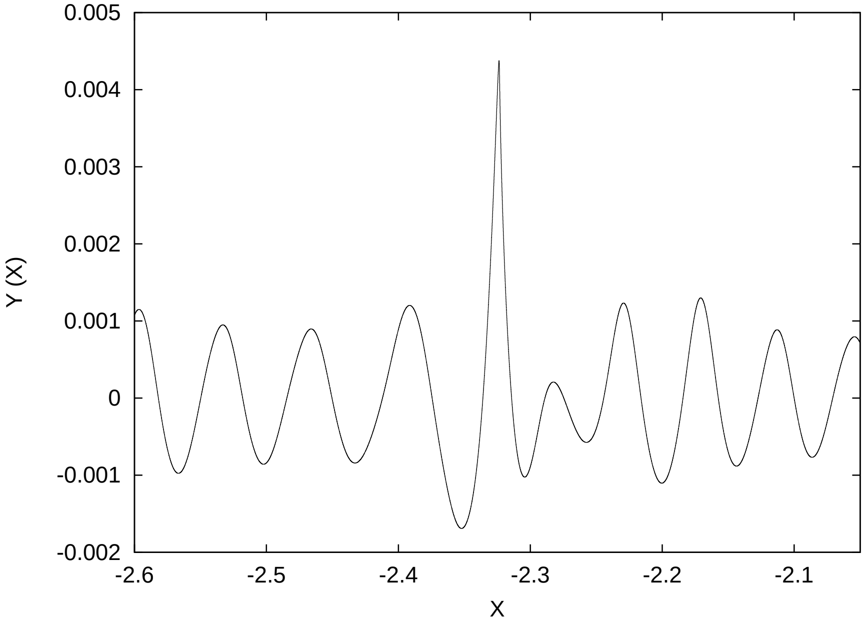


Figure 9. Zoom in surface profile at $T = 802.07$

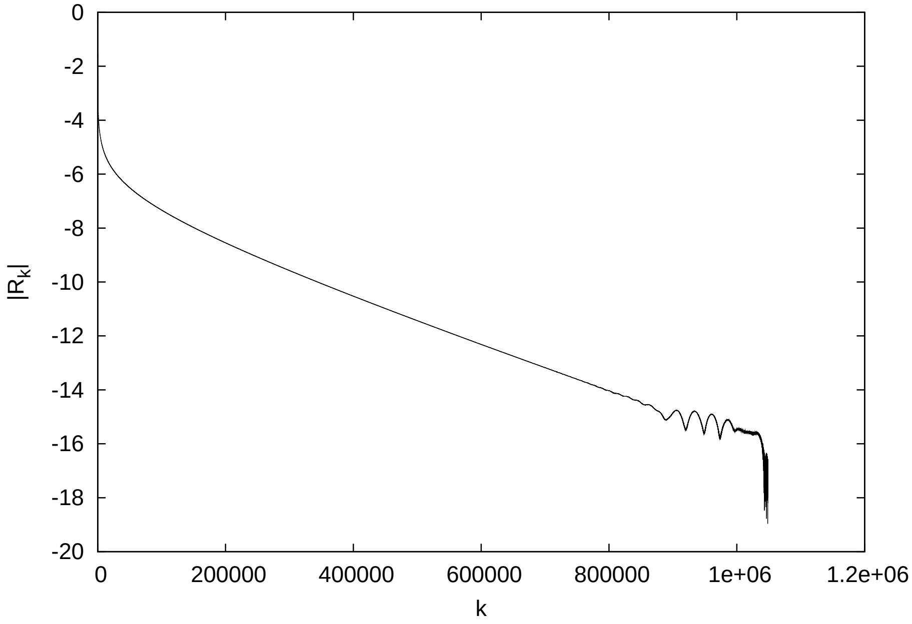


Figure 10. Fourier coefficients $|R_k|$ at $T = 802.07$.

The second term in this hierarchy was calculated by Dysthe (1979), the next one was found a few years ago by Trulsen and Dysthe (1996). The Dysthe equation was solved numerically by Ablovitz and his collaborates (see Ablovitz et al. (2000 and 2001)).

Since the first work of Smith (1976), many authors tried to explain the freak wave formation in terms of NLSE and its generalizations, like Dysthe equation.

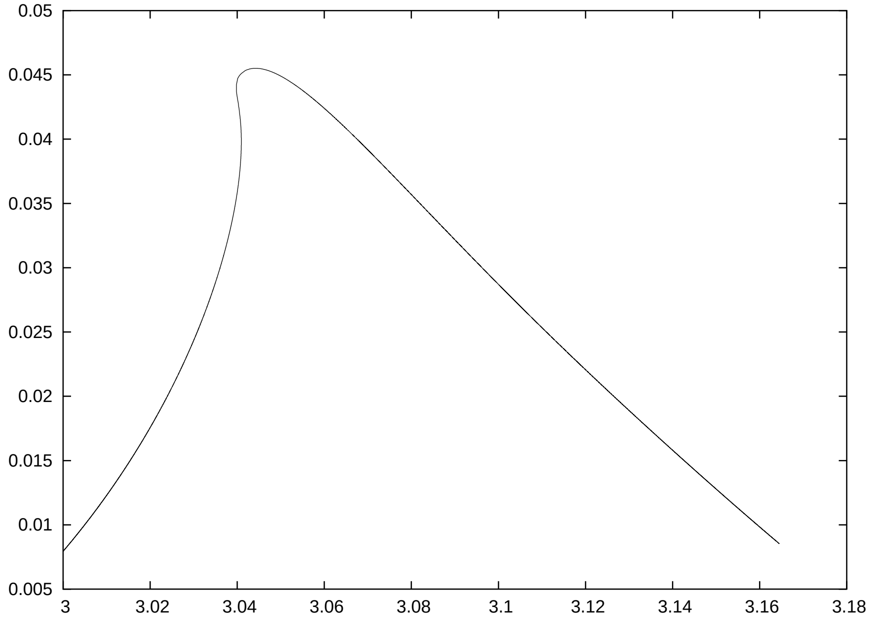


Figure 11. Profile of breaking wave.

A vast scientific literature is devoted to this subject. The list presented below is long but incomplete: Ablovitz et al. (2000 and 2001), Onorato et al. (2000a), Onorato et al. (2000b), Onorato et al. (2001), Onorato et al. (2002), Peregrine (1983), Peregrine et al. (1988), Trulsen and Dysthe (1996), Trulsen and Dysthe (1997), Trulsen (2000), Trulsen et al. (2000).

One cannot deny some advantages achieved by the use of the envelope equations. Results of many authors agree in one important point: nonlinear development of modulation instability leads to concentration of wave energy in a small spatial region. This is a "hint" regarding possible formation of freak wave. On the other hand, it is clear that the freak wave phenomenon cannot be explained in terms of envelope equations. Indeed, $NLSE$ and its generalizations are derived by expansion in series on powers of parameter $\lambda \simeq \frac{1}{Lk}$, where k is a wave number, L is a length of modulation. For real freak wave $\lambda \sim 1$ and any "slow modulation expansion" fails. At this point interesting question rises: what happens to $NLSE$ approximation when increasing the steepness of the carrier wave? In particular, we study "exact" soliton solutions for $NLSE$ placed in the exact equations (2.2).

Such type of problem was considered in the Henderson et al. (1999), but with low resolution, and small length of periodic carrier. Also in Clamond et al. (in preparation) numerical solutions for envelope equation was compared with "almost" exact equations.

Initial conditions consist of "linear wave carrier" e^{-ik_0u}, modulated in accordance with soliton solution for $NLSE$:

$$R(u) = 1 + s_0 \frac{\mathrm{e}^{-ik_0 u}}{\cosh(\lambda k_0 u)},$$
$$V(u) = -ic_0 s_0 \frac{\mathrm{e}^{-ik_0 u}}{\cosh(\lambda k_0 u)}. \tag{5.5}$$

Here s_0 is the steepness of the carrier wavetrain, c_0 - phase velocity of the carrier.

5.1 Small steepness

First experiment was intended to observe how *NLSE* works. In the initial conditions (5.5) we used

$$s_0 \simeq 0.07, \quad \lambda = 0.1, \quad k_0 = 100.$$

Initial surface of fluid is shown in Figure 12.

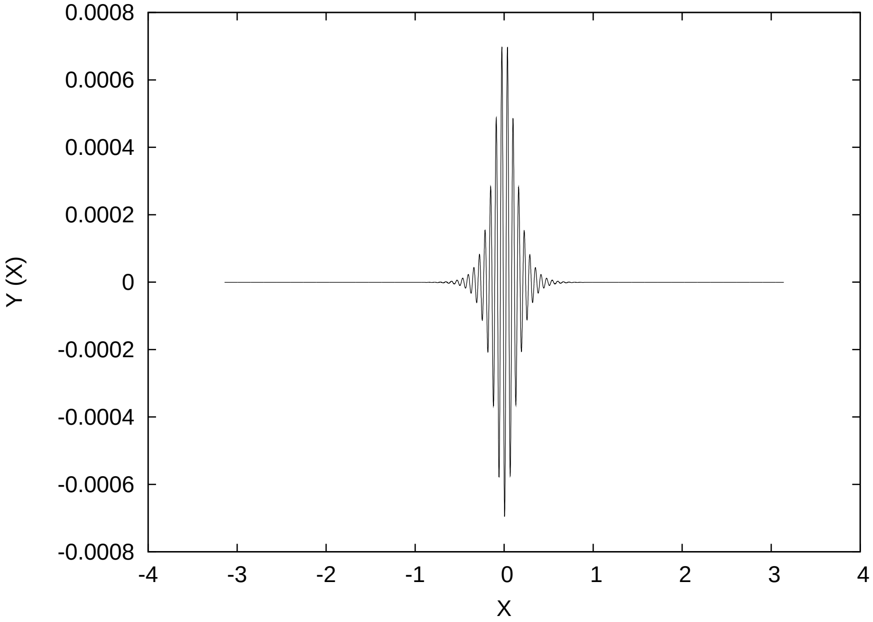


Figure 12. Initial surface profile like for *NLSE* soliton with $\mu \simeq 0.07$.

After couple of thousands wave periods soliton changes a little, as it is seen in Figure 13:

Also in the Figure 14 and Figure 15 Fourier spectra of the soliton at both moments of time are presented.

So, one can can see that for the steepness $\mu \leq 0.07$ *NLSE* model is quite reasonable.

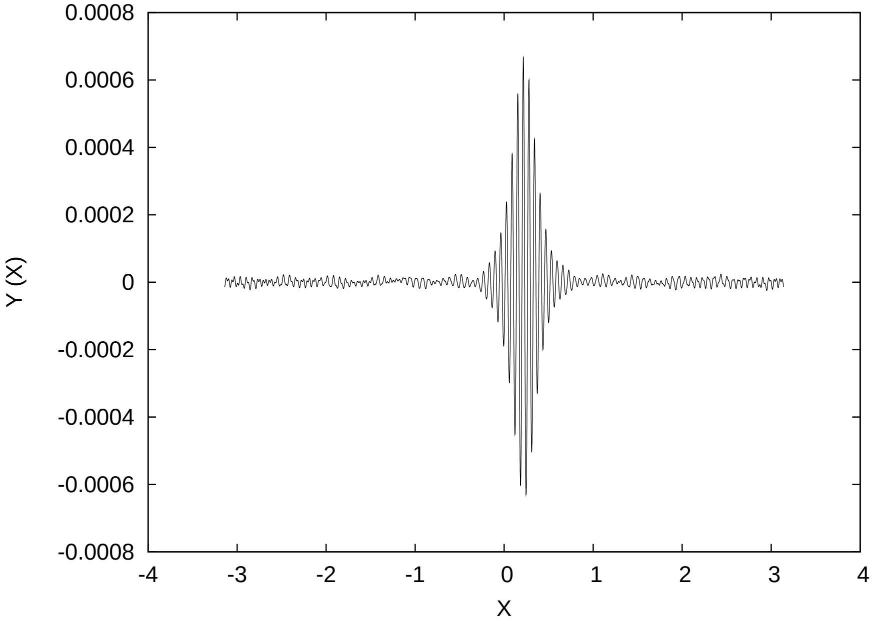


Figure 13. Surface profile like for $NLSE$ soliton with $\mu \simeq 0.07$ at T=1500.

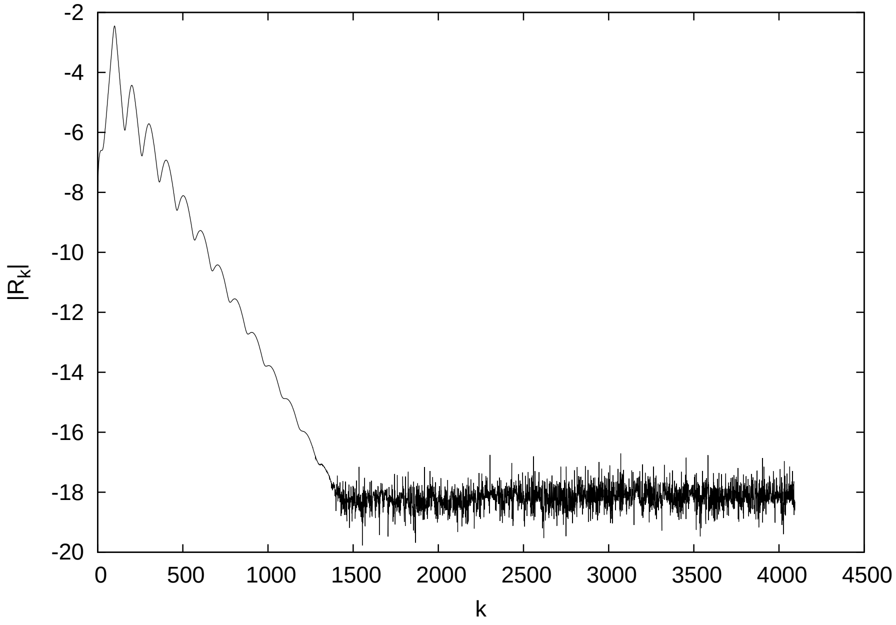


Figure 14. Fourier harmonics of the initial soliton with $\mu \simeq 0.07$.

Another numerical experiment showing effective simulation with equations (2.2) along with applicability $NLSE$ model for moderate steepness, $\mu \simeq 0.085$, is the collision of two solitons.

In the Figure 16 initial condition is shown:

Moment of collision is shown in the Figure 17:

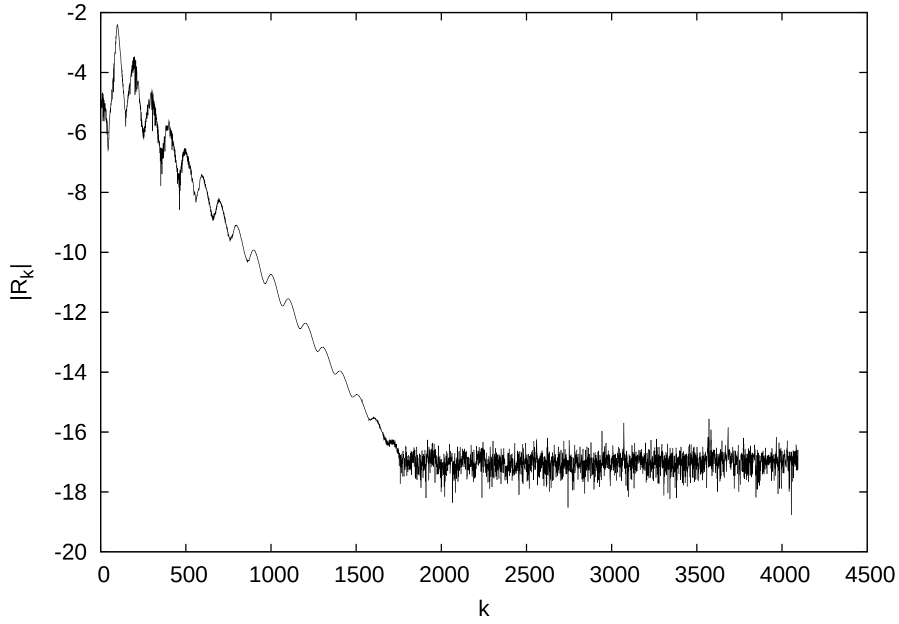


Figure 15. Fourier harmonics of the soliton with $\mu \simeq 0.07$ at T=1500.

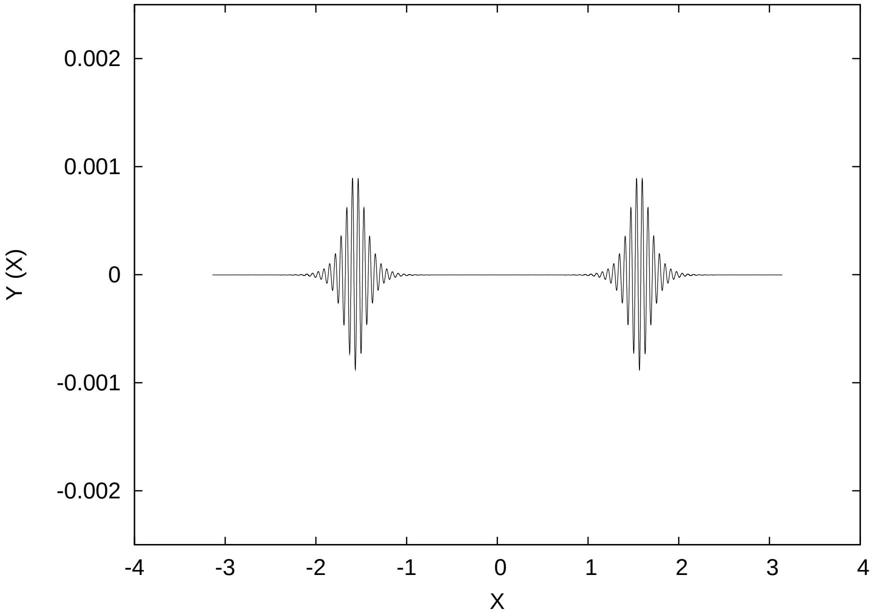


Figure 16. Initial surface profile of two $NLSE$ solitons with $\mu \simeq 0.085$.

and detailed view showing carrier wavetrain under the envelope is in the Figure 18

After second collision (recall that boundary conditions are periodic) solitons are plotted in the Figure 19:

Fourier spectra of these two solitons at the moments of time $T = 0.05, 30.8, 250.0$ are shown in Figure 20, 21 22.

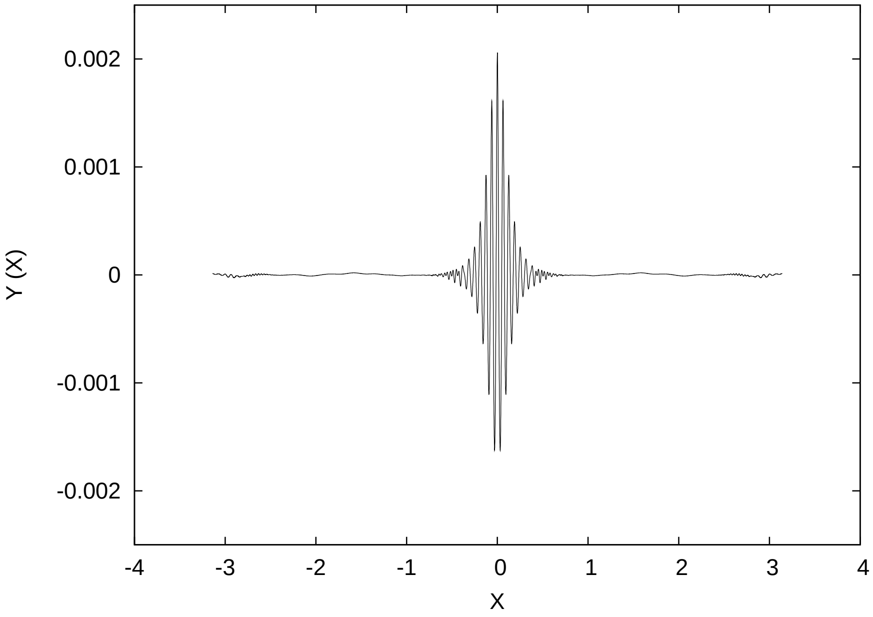


Figure 17. Two $NLSE$ solitons with $\mu \simeq 0.085$. collide at T=30.8

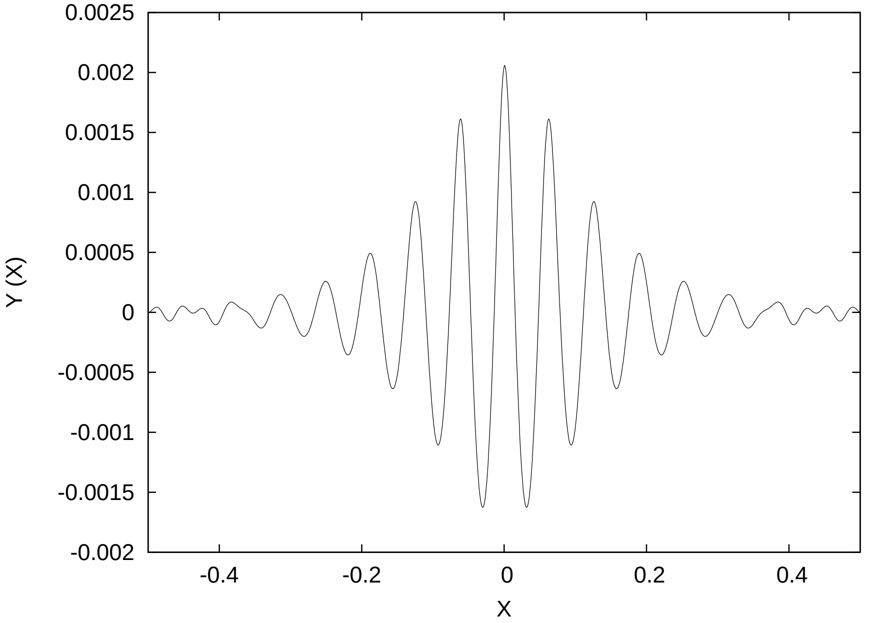


Figure 18. Detailed view of two colliding $NLSE$ solitons with $\mu \simeq 0.085$. at T=30.8

5.2 Large steepness

Now let's turn to the higher steepness of the carrier,

$$\mu = 0.1.$$

In the Figure 23 there is initial condition:

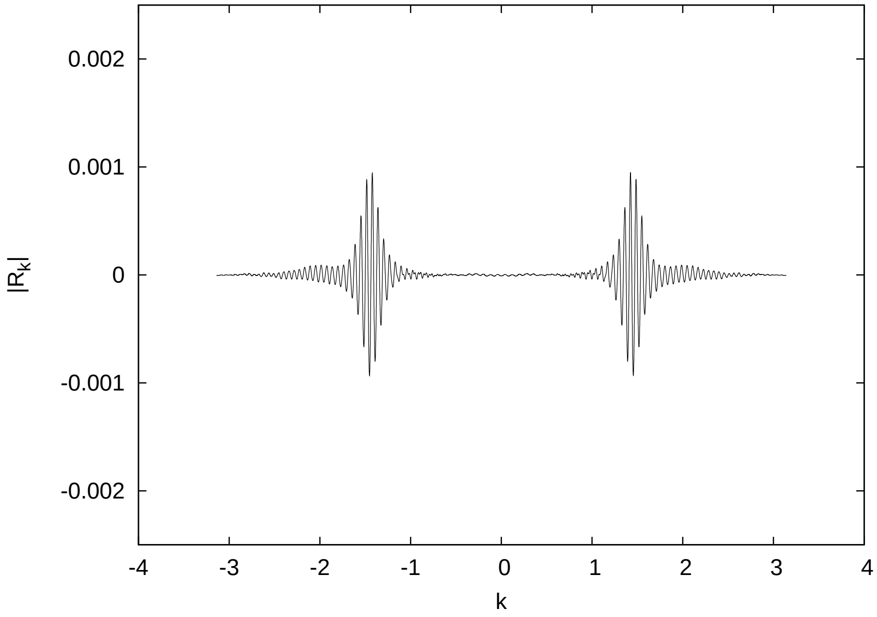


Figure 19. Two $NLSE$ solitons with $\mu \simeq 0.085$. after two collisions at T=250.0

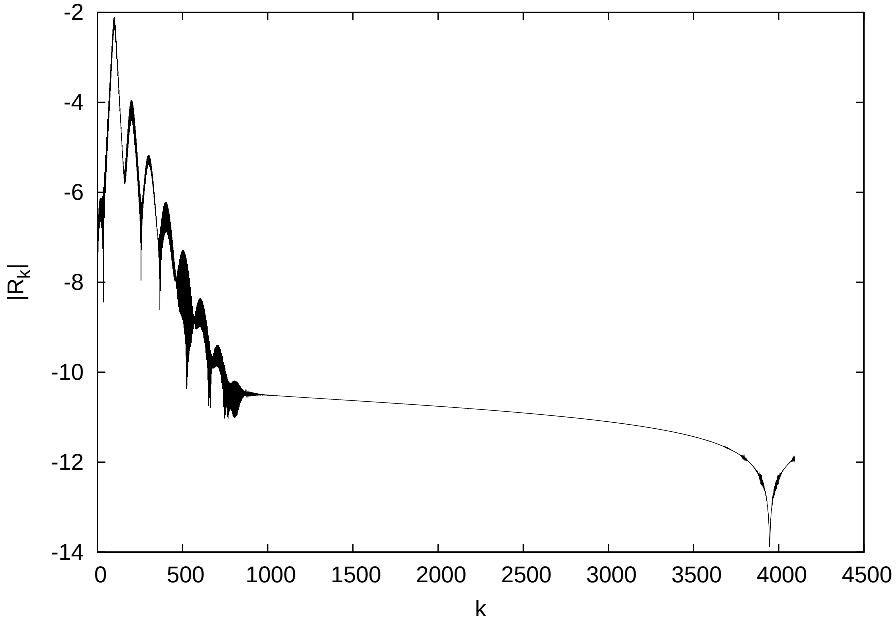


Figure 20. Fourier spectrum of the initial surface profile of two $NLSE$ solitons with $\mu \simeq 0.085$.

Again, after couple of thousands wave periods soliton changes a little, as it is seen in Figure 24:

And in the Figure 25 and Figure 26 Fourier spectra of the soliton at both moments of time are presented.

From this pictures one can see that for steepnees $\mu \simeq 0.10$ some corrections to the $NLSE$ model are desirable. Dysthe equations are exactly intended for

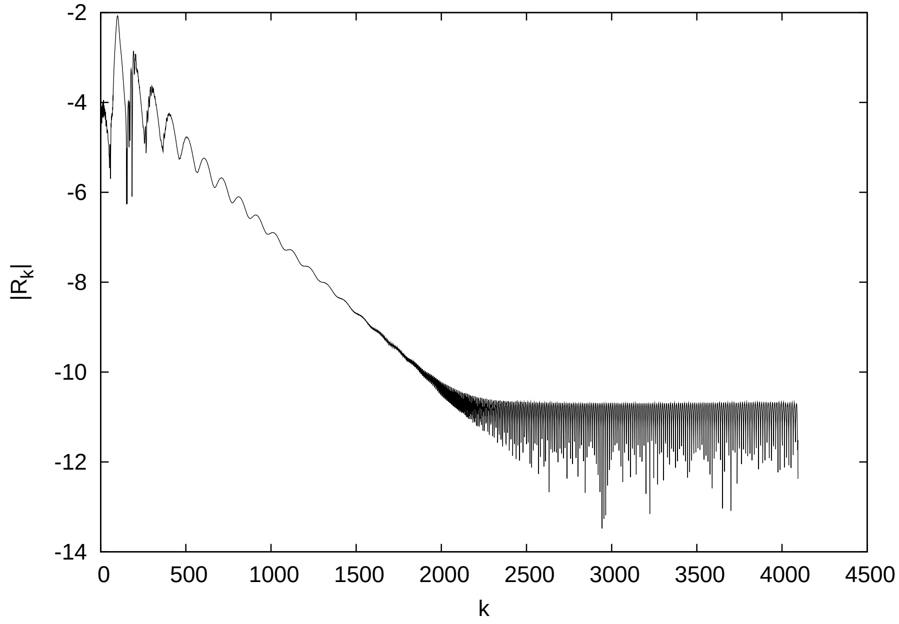


Figure 21. Fourier spectrum of two colliding $NLSE$ solitons with $\mu \simeq 0.085$. at $T = 30.8$

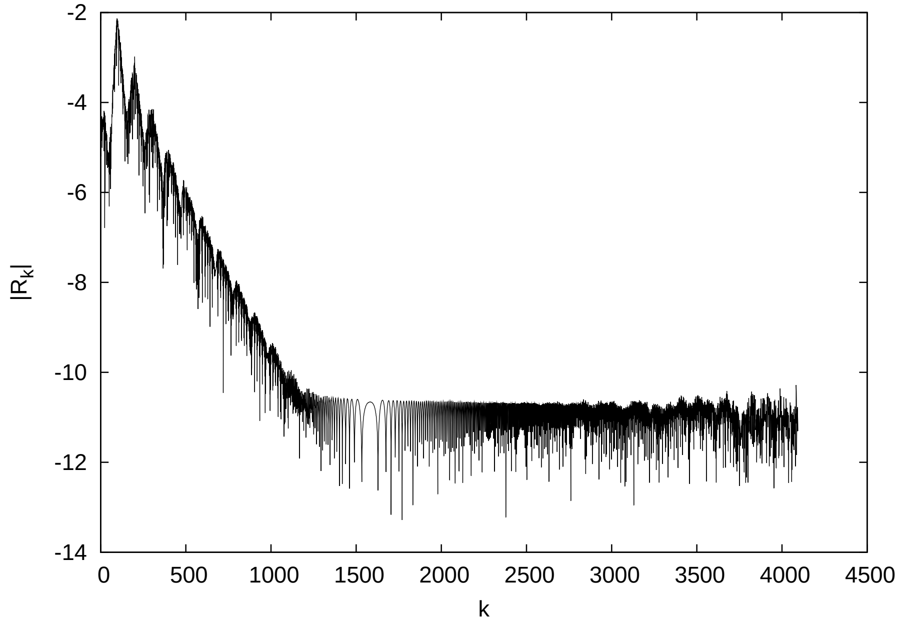


Figure 22. Fourier spectrum of two $NLSE$ solitons with $\mu \simeq 0.085$. at $T = 250.0$

that situation.

But what happens when further increasing the steepness? Below we consider the case of the steepness of the carrier

$$\mu = 0.14.$$

In the Figure 27 there is initial condition:

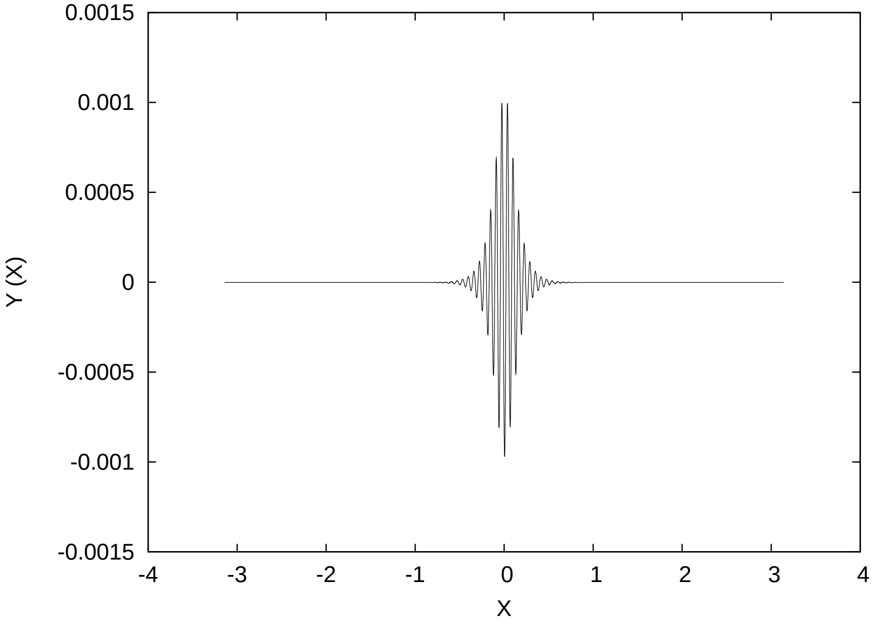


Figure 23. Initial surface profile like for $NLSE$ soliton with $\mu \simeq 0.10$.

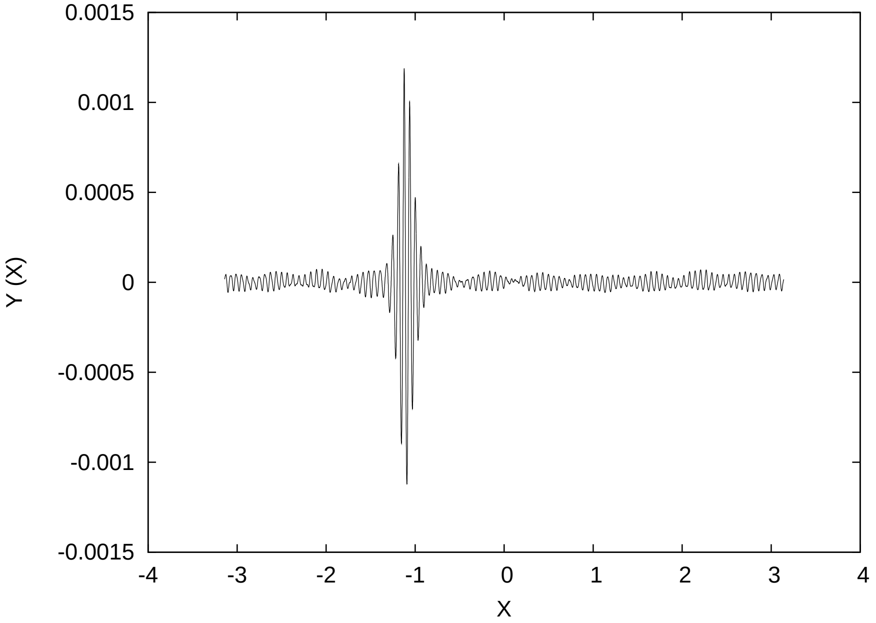


Figure 24. Surface profile like for $NLSE$ soliton with $\mu \simeq 0.10$ at T=2345.

Very fast, after couple of dozen wave periods soliton drasticaly changes, as it is seen in Figure 28:

One can see freak wave at the surface (in Figure 29):

And in the Figure 30 and Figure 31 Fourier spectra of the soliton at both moments of time are presented. They demonstrate the quality of the simulation.

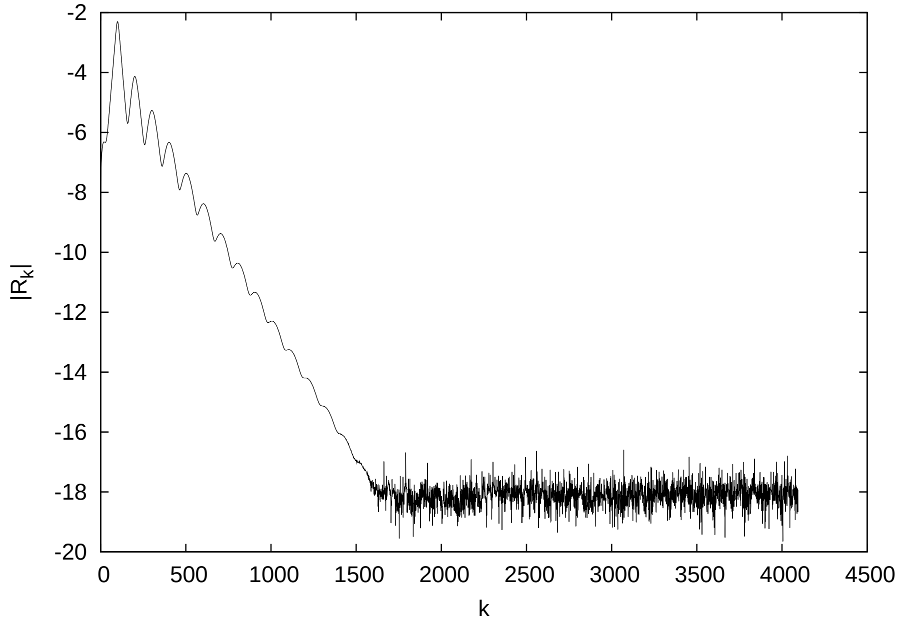


Figure 25. Fourier harmonics of the initial soliton with $\mu \simeq 0.10$.

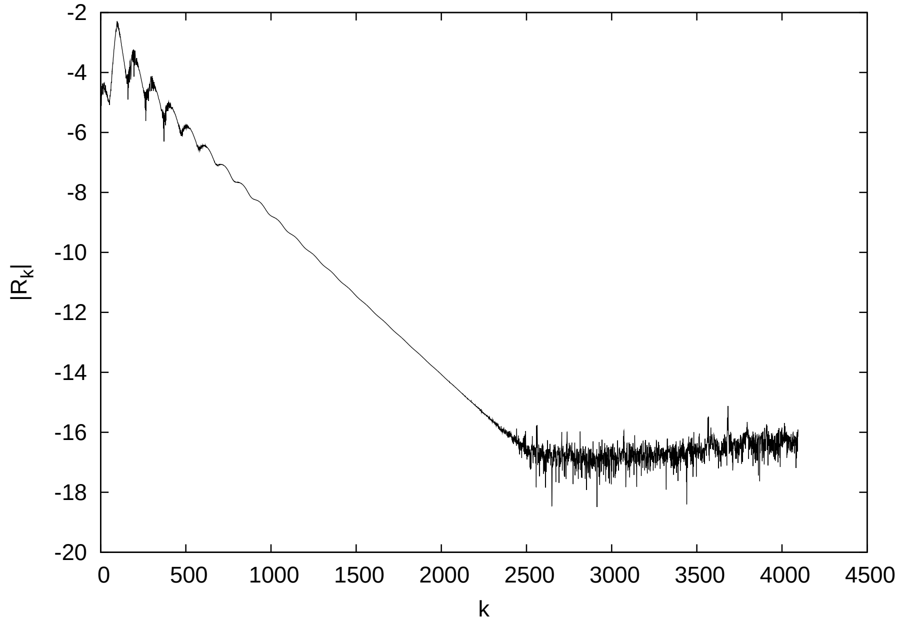


Figure 26. Fourier harmonics of the soliton with $\mu \simeq 0.10$ at T=2345.

From the last case, with the steepness $\mu = 0.14$, one can see that envelope approximation completly fails. Such event as one single crest (freak wave) can not be described in terms of wave envelope.

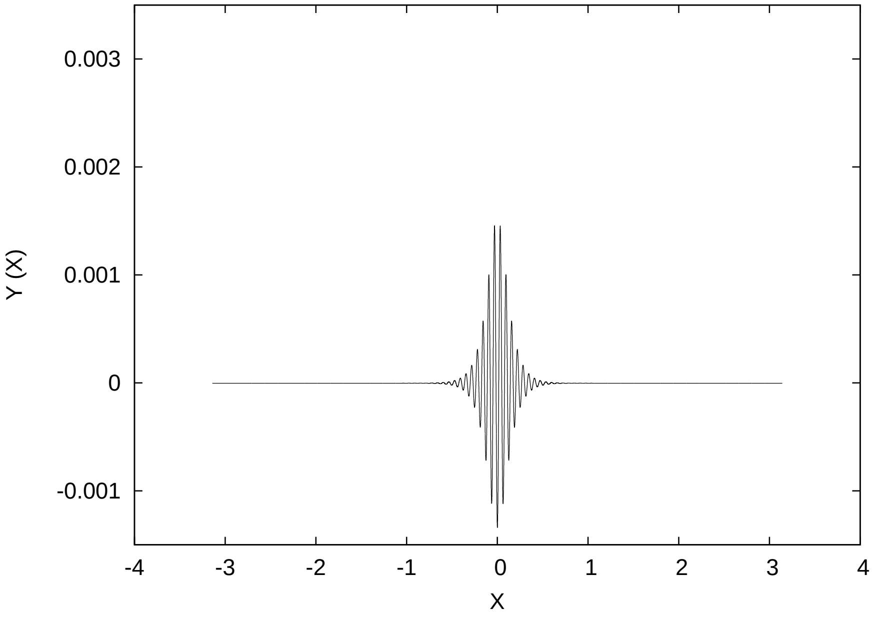


Figure 27. Initial surface profile like for $NLSE$ soliton with $\mu \simeq 0.14$.

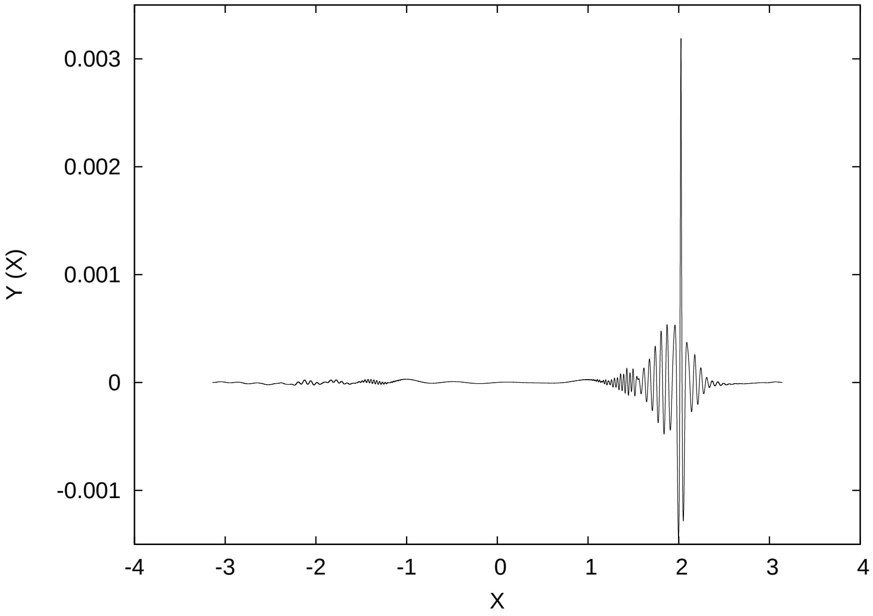


Figure 28. Surface profile like for $NLSE$ soliton with $\mu \simeq 0.14$ at T=38.4.

6 Conclusions

Let us summarize our numerical experiments. Certainly, they reproduce the most apparent features of freak waves: single wave crests of very high amplitude, exceeding the significant wave height more than three times, appearing from "nowhere" and reaching full height in a very short time, less than ten

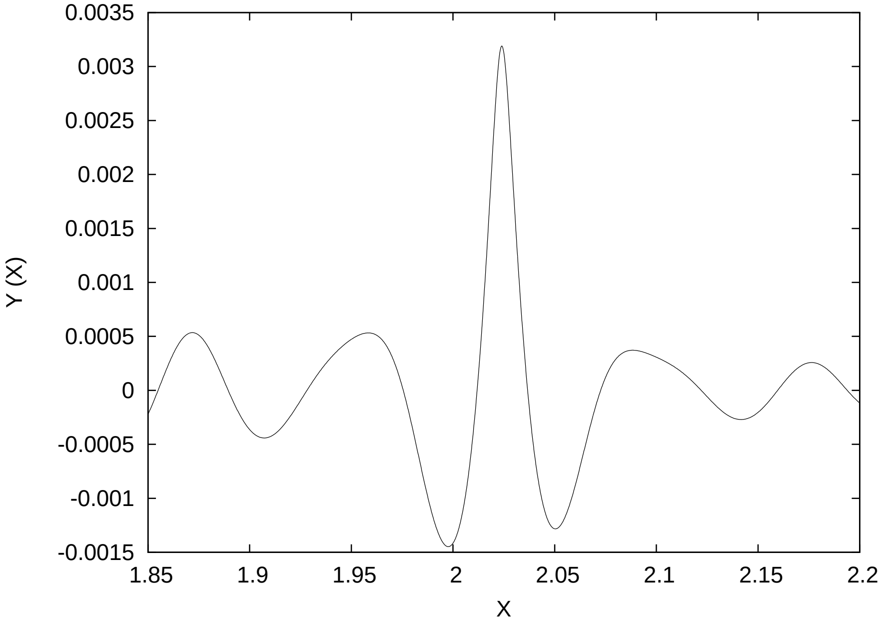


Figure 29. Zoomed surface profile near freak wave $\mu \simeq 0.14$ at T=38.4.

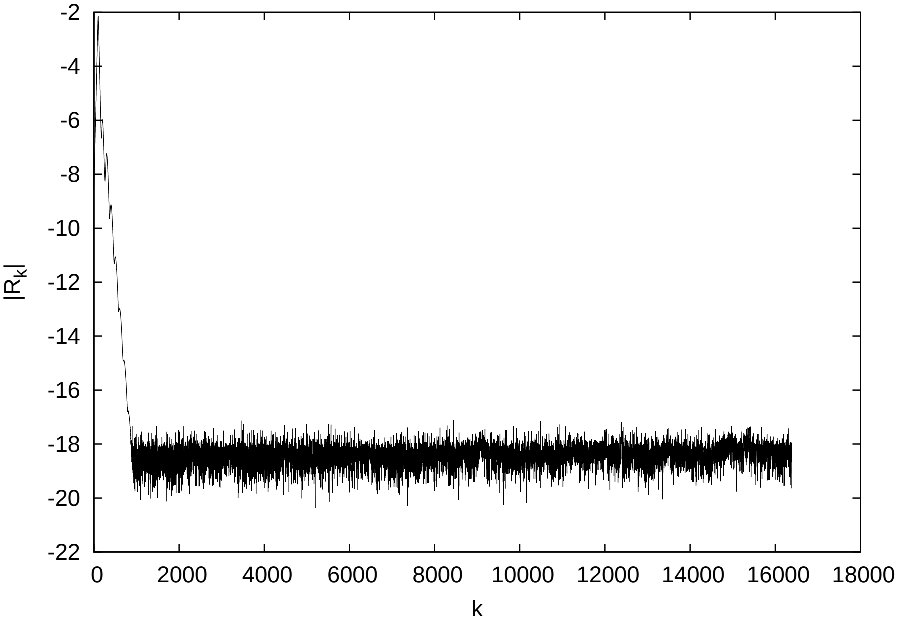


Figure 30. Fourier harmonics of the initial soliton with $\mu \simeq 0.14$.

periods of surrounding waves. The singular freak wave is proceeded by the area of diminished wave amplitudes. Final "fate" of the freak wave can be breaking. Freak wave moves with the group velocity.

In our experiments, the freak wave appears as a result of development of modulation instability (if the threshold of the instability is not exceeded no freak waves appear at all). Than it takes a long time for the onset of instability

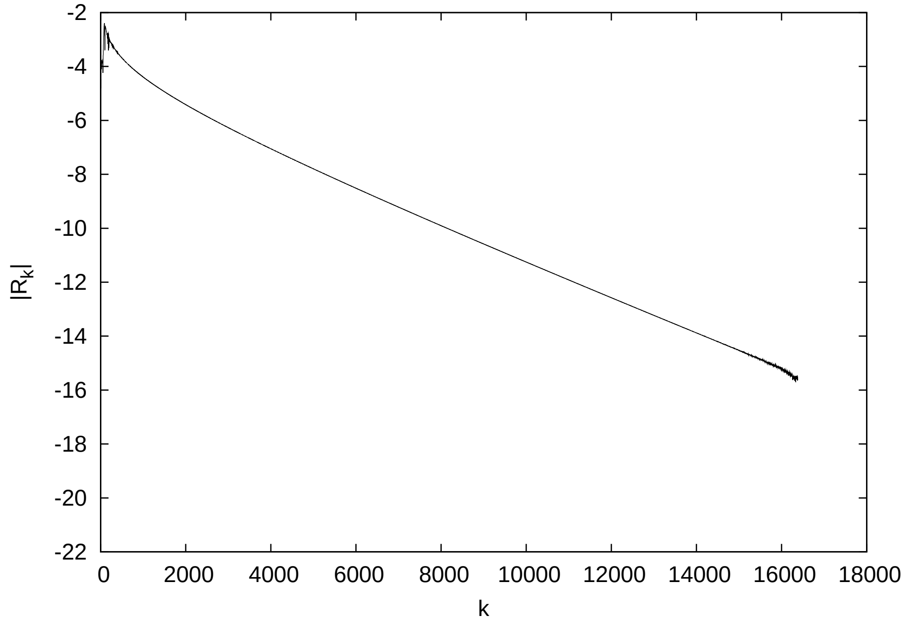


Figure 31. Fourier harmonics of the soliton with $\mu \simeq 0.14$ at T=38.4.

to create a freak wave. What happens after developing of instability but before formation of freak wave? This stage could be considered as a development of some defect on the periodic grid. This grid is just initial Stokes wave train.

Regarding $NLSE$ model for the free surface hydrodynamics we observed that if steepness is low, than the model is in good agreement with exact equations for long time. However, when increasing steepness of the carrier wave train, envelope approximation compleatly breaks.

Acknowledgments

This work was supported by the US Army Corps of Engeneers Grant W912HZ-05-P-0351, by ONR Grant N00014-03-1-0648, by NSF Grant DMS 0072803, by RFBR Grant 03-01-00289, by the Program "Mathematical Problems in Nonlinear Dynamics" from the RAS Presidium, by the Grant "Leading Scientific Schools of Russia".

We use this oppotunity to greatfully acknowledge the support of these foundations.

References

Ablovitz, M.I., Hammack, D., Henderson, J. and Scholder, C.M., Phys. Rev. Lett. (2000), 887-890;

Ablovitz, M.I., Hammack, D., Henderson, J. and Scholder, C.M., Physica D 152-153 (2001), 416-433.

Benjamin, T.B. and Feir, J.E., J. Fluid Mech. 27 (1967), 417-430.

Chase, G.A., Big wave, 2003, http://bell.mma.edu/ achase/NS-221-Big-Wave.html

Clamond, D., Francius.M., Grue. J. and Kharif. C., J. Fluid Mech. (in preparation)

Dean, R.G., Freak Waves: A possible explanation. In *Water Wave Kinetics* (ed. A. Torum and O. T. Gudmestad), Kluwer, 1990, pp. 609-612.

Divinsky, V.D., Levin, B.V., Lopatikhin, L.I., Pelinovsky, E.N. and Slyunyaev, A.V., Doklady Earth Science, Vol. 395 (2004), 438-443.

Dyachenko, A.I., E. Kuznetsov, E.A., Spector, M.D. and Zakharov, V.E., Phys. Lett. A, 221 (1996), 73-79.

Dyachenko, A.I., Zakharov, V.E. and Kuznetsov, E.A., Plasma Physics Reports, 22 # 10 (1996), 829-840.

Dyachenko, A.I. Doklady Mathematics, 63 (2001), 115-118.

Dyachenko, A.I. "Instability in the Numeric Models of Free Surface Hydrodynamics", International symposium "TOPICAL PROBLEMS OF NONLINEAR WAVE PHYSICS (NWP-2005), Sankt-Peterburg - Nizhnii Novgorod, Russia, 2-9 August, 2005.

Dyachenko, A.I. and Zakharov, V.E., JETP Letters, 81, (2005), 255-259.

Dysthe, K.B., Proc. Roy. Ser. A 369 (1979), 105-114.

Earle, M.D., J. Geophys. Res. 80 (1975), 377-379.

Feir, J.E., Proc. R. Soc. Lond. A 299 (1967), 54-58.

Gerber, M., J. Fluid Mech. 176 (1987) 311-332.

Gerber, M., J. Fluid Mech. 248 (1993), 153-172.

Gerber, M., Deep sea Res. 1996.

Gutshabash, Y.S. and Lavrenov, I.V., Izv. Atmos. Ocean Phys. 22 (1986) 494-497.

Henderson, K.L. D., Peregrine, D.H. and Dold, J.W. Wave Motion, 29 (1999), 341-361.

Irvine, D.E., Tilley, D.G., J. Geophys. Res. 93 (1988), 15389-15401.

Kharif, C., and Pelinovsky, E., Europ. J. Mech. B/Fluids, 22 (2003), 603-634.

Lavrenov, I.V., Natural Hazards 17 (1998), 117-127.

Lighthill, M.J., J. Inst. Math. Appl. 1 (1965), 269-306.

Mallory, J.K., Inst. Hydrog. Rev. 51 (1974), 89-129.

Mori, N., Liu, P.C.,and Yasuda, T., Ocean Engrg. 29 (2002), 1399-1414.

Onorato, M., Osborne, A.R., Serio, M. and Damiani, T., Occurence of freak waves from envelope equation in random ocean wave simulations, in *Rogue waves 2000: Brest, France, November 2000*, eds. M. Olagnon and G.A. Athanassoulis, Ifremer, (2001), 181-192.

Onorato, M., Osborne, A.R. and Serio, M., Phys. Lett. A 275 (2000), 386-393.

Onorato, M., Osborne, A.R., Serio, M. and Bertone, S., Phys. Rev. Letters 86 (2001), 5831-5834.

Onorato, M., Osborne, A.R. and Serio, M., Phys. of Fluids, 14 (2002), L25-L28.

Peregrine, D.H., Adv. Appl. Mech. 16 (1976), 9-117.

Peregrine, D.H., J. Austral. Math. Soc. B 25 (1983), 16-43.

Peregrine, D.H., Skyner, D., Stiassnie, M. and Dold, N., *Proc. 21th Intl. Conf. on Coastal Engng.* Vol. 1, Chap. 54, (1988), pp. 732-742.

Slunyaev.A., Kharif. C., Pelinovsky. E. and Talipova. T., Physica D, 173 (2002), 77-96.

Ruban, V.P. Phys. Rev. E, 71 (2005), 055303(R).

Sand, S.E., Hansen, N.E., Klinting, P., Gudmestad, O.T., Sterndorff, M.J., Freak wave kinematics, in Water Wave Kinematics, eds. A. Torum, O. T. Gudmestad, Kluwer Academic, Doedrecht, (1990), 535-549.

Smith, R., Giant Waves, Fluid Mech. 77 (1976), 417-431.

Song, J. and Banner, M.L., J. Phys. Oceanogr. 32 (2002), 2541.

Trulsen, K. and Dysthe, K.B., Waves (1996), 281-289.

Trulsen, K. and Dysthe, K.B., Freak waves - a three-dimensional wave simulation, in Proc. 21st Symposium on Naval Hydrodynamics, 1997, 550-558; http://www.nap.edu/books/0309058791/html/550.html

Trulsen, K., in *Rogue waves 2000: Brest, France, November 2000*, eds. M. Olagnon and G.A. Athanassoulis, Ifremer, 2001, 265-274.

Trulsen, K., Kliakhandler, I., Dysthe, K.B. and Velarde, M.G. Phys. Fluids 12, (2000) 2432-2437.

White, B.S., Fornberg, B., J. Fluid Mech. 355 (1998), 113-138.

Zakharov, V.E., J. Teor. Prikl. Fiz. 51 (1966), 668-671 (in Russian); Sov.Phys. JETP 24 (1967), 455-459.

Zakharov, V.E., J. Appl. Mech. Tech. Phys. 9 (1968), 190-194.

Zakharov, V.E. and Shabat, A.B., Soviet Physics JETP 34 (1972), 62-69.

Zakharov, V.E., Amer. Math. Soc. Ser. 2, 182 (1998), 167-197.

Zakharov, V.E., Dyachenko A.I. and Vasilyev, O.A., Eur. J. of Mech. B-Fluids, 21 (2002), 283-291.