Nonlinear three dimensional water waves

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1 Introduction

Numerical solutions for three-dimensional gravity capillary waves in water of finite depth are presented. The full Euler equations are used and the waves are calculated by boundary integral equation methods. The results presented below are based on the following papers, where more details can be found

- E. Părău and J.-M. Vanden-Broeck, Nonlinear two- and three- dimensional free surface flows due to moving disturbances, *Eur. J. Mechanics B/Fluids*, **21**, 6 (2002), 643-656,
- E. Părău, J.-M. Vanden-Broeck and M. J. Cooker, Nonlinear three-dimensional gravity-capillary solitary waves, *J. Fluid Mechanics*, **536** (2005), 99-105,
- E. Părău, J.-M. Vanden-Broeck and M.J.Cooker, Three-dimensional gravity-capillary solitary waves in water of finite depth and related problems, *Physics of Fluids*, **17**,12 (2005), 122101, 9p,
- E. Părău, J.-M. Vanden-Broeck and M.J.Cooker, Three-dimensional gravity and gravity-capillary interfacial flows, submitted to *Mathematics and Computers in Simulation*, 2005.

2 Formulation

The fluid is incompressible and the flow is irrotational. We are interested in steady waves which travel at a constant velocity c on water of finite depth h, and we choose a frame of reference moving with this wave speed. We introduce cartesian coordinates x, y, z with the z-axis directed vertically upwards and the x-axis in the direction of wave propagation. We denote by $z = \zeta(x, y)$ the equation of the free surface. Dimensionless variables are introduced by taking the unit length to be $T/\rho c^2$ and the unit velocity to be c, where T is the constant coefficient of surface tension, and ρ is the fluid density. In terms of the fluid velocity potential $\Phi(x, y, z)$, the problem is formulated as follows:

$$\nabla^2\Phi = 0, \qquad x, y \in \mathbf{R}, z < \zeta(x, y), \tag{1}$$

with the two surface boundary conditions

$$\Phi_x\zeta_x + \Phi_y\zeta_y = \Phi_z, \quad \text{on } z = \zeta(x, y), \tag{2}$$

$$\frac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) + \lambda\beta\zeta - \left[\frac{\zeta_x}{\sqrt{1 + \zeta_x^2 + \zeta_y^2}}\right]_x - \left[\frac{\zeta_y}{\sqrt{1 + \zeta_x^2 + \zeta_y^2}}\right]_y = \frac{1}{2}, \text{ on } z = \zeta(x, y), \tag{3}$$

$$\Phi_z = 0, \quad \text{on } z = -\frac{1}{\beta}, \tag{4}$$

which is the boundary condition on the horizontal impermeable bed, where

$$\lambda = \frac{gh}{c^2} \qquad \text{and } \beta = \frac{T}{\rho h c^2}.$$

As we are looking for fully localised three-dimensional solitary waves, we use the conditions

$$(\Phi_x, \Phi_y, \Phi_z) \rightarrow (1,0,0),\ \zeta \rightarrow 0,\ \text{as} \quad (x^2+y^2)^{1/2} \rightarrow \infty \tag{5}$$

in order to fix the value of Bernoulli's constant on the right-side of equation (3).

3 The method

The formulation involves applying Green's second identity to the functions $\Gamma = \Phi - x$ and G where G is the three dimensional free space Green function

$$G = \frac{1}{4\pi} \frac{1}{((x-x^*)^2 + (y-y^*)^2 + (z-z^*)^2)^{1/2}}, \tag{6}$$

for a region V which consists of a cylinder bounded by the free surface (except a small hemisphere around the point $Q(x^*, y^*, z^*)$), and its image $S_{F'}$ on the other side of the bottom $z = -1/\beta$. In this way we satisfy (4) as a symetry property.

After projecting the surface integrals onto the Oxy plane, we obtain

$$\frac{1}{2}(\phi(x^*,y^*) - x^*) =$$

$$= \int\!\!\int_{\mathbf{R}^2} (\phi(x,y)-x) \frac{1}{4\pi} \frac{\zeta(x,y) - \zeta(x^*,y^*) - (x-x^*)\zeta_x(x,y) - (y-y^*)\zeta_y(x,y)}{((x-x^*)^2 + (y-y^*)^2 + [\zeta(x,y)-\zeta(x^*,y^*)]^2)^{3/2}} dxdy +$$

$$+ \int\!\!\int_{\mathbf{R}^2} \frac{1}{4\pi} \frac{\zeta_x}{((x-x^*)^2 + (y-y^*)^2 + [\zeta(x,y)-\zeta(x^*,y^*)]^2)^{1/2}} dxdy$$

$$+ \int\!\!\int_{\mathbf{R}^2} (\phi(x,y)-x) \frac{1}{4\pi} \frac{\zeta(x,y) + \zeta(x^*,y^*) + 2/\beta - (x-x^*)\zeta_x - (y-y^*)\zeta_y}{((x-x^*)^2 + (y-y^*)^2 + [\zeta(x,y)+\zeta(x^*,y^*)+2/\beta]^2)^{3/2}} dxdy +$$

$$+ \int\!\!\int_{\mathbf{R}^2} \frac{1}{4\pi} \frac{\zeta_x}{((x-x^*)^2 + (y-y^*)^2 + [\zeta(x,y)+\zeta(x^*,y^*)+2/\beta]^2)^{1/2}} dxdy \tag{7}$$

where $\phi(x,y) = \Phi(x,y,\zeta(x,y))$.

The singularities in the integrands may be isolated in one term by addition and subtraction of a null quantity, which can be evaluated in closed form (see Forbes (1989) for details).

The solution is symmetric with respect to both x and y, so we only compute one quarter-plane. We truncate the intervals $0 < x < \infty$ and $0 < y < \infty$ to $x_1 < x < x_N$, and $y_1 < y < y_M$ and introduce the mesh points $x_i = (i-1)\Delta x$, $i = 1, \ldots, N$ and $y_j = (j-1)\Delta y$, $j = 1, \ldots, M$. The $2NM$ unknowns are

$$\mathbf{u} = (\zeta_{x_{11}}, \zeta_{x_{12}}, \ldots, \zeta_{x_{N,M-1}}, \zeta_{x_{NM}}, \phi_{11}, \ldots, \phi_{NM})^T,$$

where we use the notation $\zeta_{x_{ij}} = \zeta_x(x_i, y_j)$, $\phi_{ij} = \phi(x_i, y_j)$, etc.

The integrals and the Bernoulli equation are evaluated at the points $(x_{i+1/2}, y_j)$, $i = 1, ..., N-2$, $j = 1, ..., M$ so we have $2(N-2)M$ equations. Another $2M$ equations are obtained from the truncation conditions $\zeta_{x_{Nj}} = 0$, $\phi_{x_{Nj}} = 1$, $j = 1, \ldots, M$ and another $2M$ equations are given by the symmetry conditions $\zeta_{x_{1j}} = 0$ and by $\phi_{1j} = 0$.

The $2NM$ nonlinear equations are solved by Newton's method.

4 Results

For small surface tension ($\beta < 1/3$) we found that the three dimensional problem is qualitatively similar to the two dimensional problem (see Vanden-Broeck and Dias (1992), Dias et al. (1996)). In particular there are branches of fully localised depression or elevation three-dimensional gravity-capillary solitary waves. These waves have decaying oscillations in the direction of propagation and are monotonically decaying perpendicular to the direction of propagation, as shown in Figures 1 and 2. We can compute waves of this type when the depth of the water becomes infinite (see Figure 3).

For strong surface tension ($\beta > 1/3$) we found only fully localised depression gravity capillary solitary waves. They have a similar form with fully localised solitary-wave solutions of the KP-I equation (see Figures 4 and 5). The agreement between the amplitude of the KP-I solution and our solution of the full equations is presented in Figure 6.

Theoretical results concerning three-dimensional fully localised solitary-wave solutions of the three-dimensional gravity-capillary water-wave problem have been obtained recently by Kim and Akylas (2005) and Milewski (2005) who derived weakly nonlinear models, and by Groves and Sun (2004) who proved rigorously their existence.

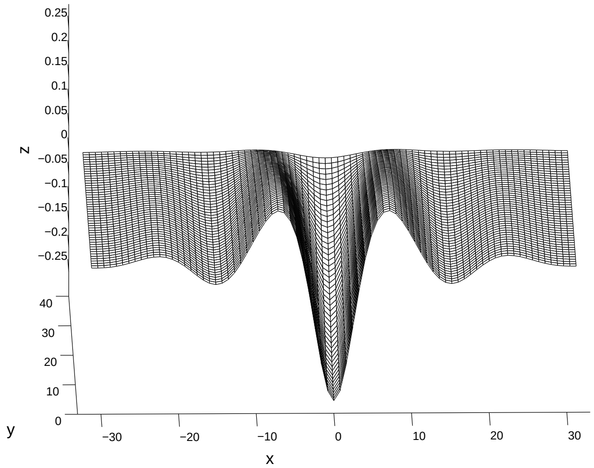

Figure 1: Central depression solitary gravity-capillary wave for $\beta = 0.235, \lambda = 1.13$. Only half of the solution ($y \geq 0$) is shown. The wave propagation direction is along the x-axis.

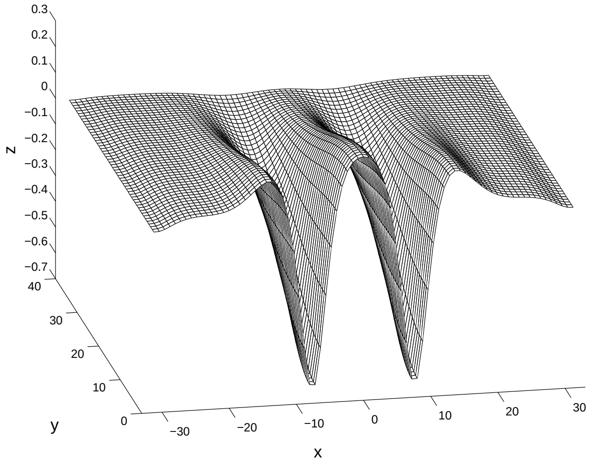

Figure 2: Central elevation solitary gravity-capillary wave for $\beta = 0.28, \lambda = 1.13$. Only half of the solution ($y \geq 0$) is shown.

References

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[2] L.K. Forbes, "An algorithm for 3-dimensional free-surface problems in hydrodynamics," J. Comp. Physics **82**, 330 (1989).

[3] M.D. Groves and S.M. Sun, "Fully localised solitary-wave solutions of the three-dimensional gravity-capillary water-wave problem," preprint (2004).

[4] B. Kim and T.R. Akylas, "On gravity-capillary lumps," J. Fluid Mech. **540**, 337 (2005).

[5] P.A. Milewski, "Thee-dimensional localized solitary gravity-capillary waves," Commun. Math. Sci. **3**, 89 (2005)

[6] J.-M. Vanden-Broeck and F. Dias, "Gravity-capillary solitary waves in water of infinite depth and related free-surface flows," J. Fluid Mech. **240**, 549 (1992).

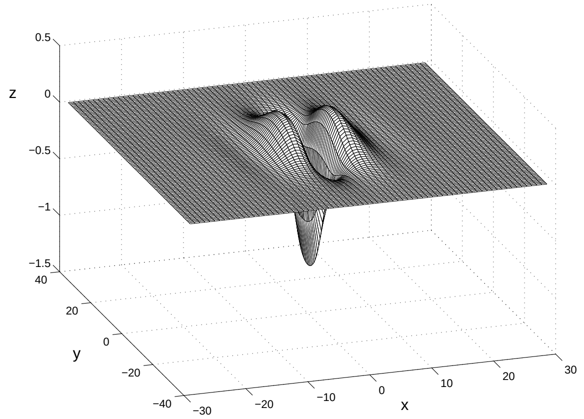

Figure 3: Solitary gravity-capillary wave in infinite depth. The wave propagates along the x-axis.

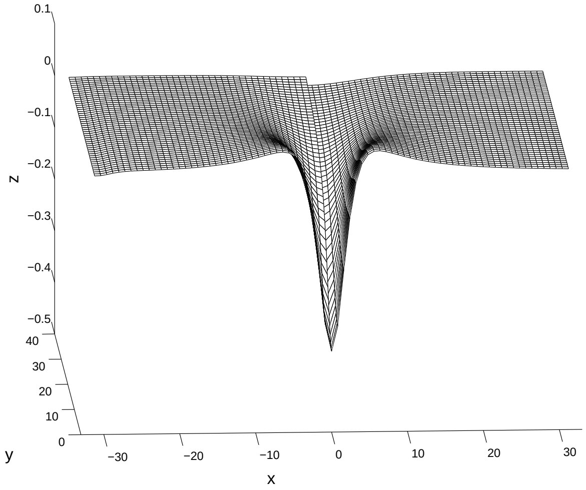

Figure 4: Central depression solitary gravity-capillary wave for $\beta = 1, \lambda = 1.14$ ($x \leq 0$) compared with the fully localised solitary wave solution for KP-I ($x \geq 0$). Only half of the solutions ($y \geq 0$) are shown.

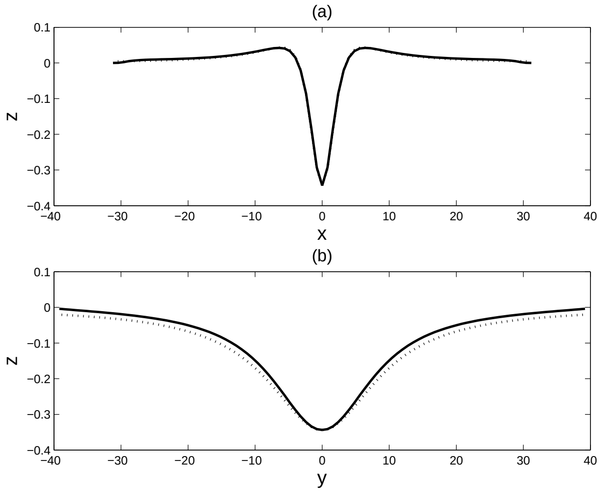

Figure 5: (a) The centreline in the Ox direction of the computed solitary gravity-capillary wave (solid line) and the fully localised solitary wave solution for KP-I (dotted line). The parameters are $\beta = 1, \lambda = 1.14$. (b) The centreline in the Oy direction of the computed solitary gravity-capillary wave (solid line) and the fully localised solitary wave solution for KP-I (dotted line). The parameters are $\beta = 1, \lambda = 1.14$.

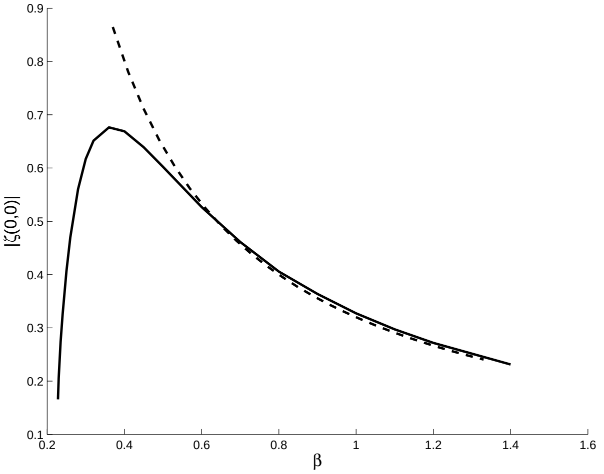

Figure 6: The maximum amplitude of the computed solution (solid line) and KP-I solutions (dashed line) for a fixed $\lambda = 0.132$.