Interaction of shallow-water solitons as a possible model for freak waves

Tarmo Soomere & Jüri Engelbrecht

Institute of Cybernetics, Tallinn University of Technology
Akadeemia tee 21, 12618 Tallinn, Estonia, http://www.ioc.ee/index.php.en
soomere@cs.ioc.ee, je@ioc.ee

Abstract. Nonlinear interactions of solitonic waves in the framework of the Kadomtsev-Petviashvili equation may result in particularly high wave humps resembling the phenomena occurring during the Mach reflection of solitary waves. For the limiting case of interactions of perfect solitons the extreme heights, slopes and many other properties of these humps can be estimated analytically. Surface elevation up to four times as high as the amplitude of the counterparts occurs if their amplitudes, the angle between their crests and the water depth are specifically balanced. The slope of the front of the high hump may be eight times as large as the maximum slope of the fronts of the interacting waves. Although such a balance occurs seldom, the resulting structure may persist for a long time until the balance is violated. This mechanism does not occur in deep water by it may be a generic source of abnormally high and steep waves in areas of moderate depth.

Solitonic shallow-water waves and their interactions on water surface

The idea that an appropriate nonlinear mechanism could be responsible for extreme waves has been employed in many studies (see overview of the relevant ideas in [1]). Perhaps the simplest mechanism able to create considerable changes in the wave amplitudes is nonlinear interaction of shallow-water solitons. The most well-known example of such structures are Korteweg-de Vries (KdV) solitons. Interaction of unidirectional KdV solitons does not create any drastic increase of surface elevation. The situation is different in two-dimensional case where KdV solitons may propagate in any direction, and may meet analogous structures propagating in other directions.

The structures arising when two KdV solitons propagating in slightly different directions meet each other can be described by means of multi-soliton solutions to the Kadomtsev-Petviashvili (KP) equation [2]. This equation was derived initially to examine small effects in a direction perpendicular to the propagation direction would have on a KdV soliton in a plasma, and only later it was shown to describe phenomena on water surface. The analytical two-soliton solution was first found by Zakharov and Shabat in [3] and generalised to the case of N solitons in [4].

V.E. Zakharov told a nice story about onset of the analytical solutions to the KP equation during the Edinburgh meeting. When he together with A.B. Shabat first published the two-soliton solution [3], it was not mentioned that the solution was only valid for a limited set of parameters of interacting waves (cf. [5]). A few years later A.C. Newell and L.G. Redekopp noticed that [6] but undiplomatically formulated the gap as a 'breakdown of Zakharov-Shabat theory'. As a result, V.E. Zakharov was accused for damaging the reputation of the otherwise perfect Soviet science. Fortunately, the obligation of having a serious word with V.E. Zakharov and to decide about the punishment (which could have been quite serious this time in the USSR!) was put on B.B. Kadomtsev. He was a brilliant scientist himself and did not take the criticism seriously. V.E. Zakharov and A.C. Newell became close friends later.

Abnormally high waves occurring owing to nonlinear interaction of nonlinear surface waves were probably first noticed in coastal engineering in 1950s [7] for the particular case of interaction of an incident wave with its reflection from a vertical wall. A part of such situations can be also described in terms of two-soliton solutions of the KP equation, surface elevation in which is symmetrical with respect of a specific point, corresponding to the maximum elevation of the water along the wall. J.W. Miles [8] demonstrated that a drastic increase in surface wave amplitude in such situations may occur in the framework of Boussinesq solitons – a more general class of solitons on water surface compared to the KP solitons. The crests of the incident and the reflected wave merge into one structure, the height of which may be four time as high as the heights of the counterparts. The resulting structure resembles Mach stem or Mach reflection, the length of which may be finite in the KP and Boussinesq frameworks but is believed to gradually increase in the case of analogous reflection of Stokes waves [9].

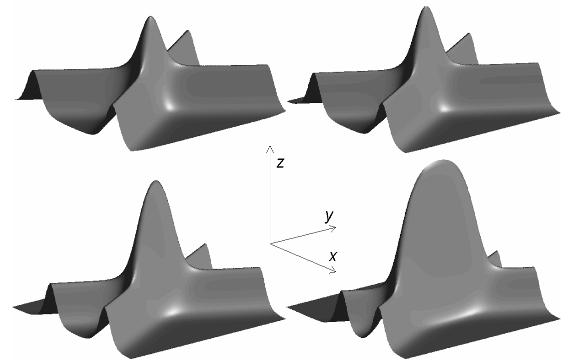

Figure 1. Surface elevation owing to interaction of equal amplitude solitons with different angles between their crests.

Since then, many studies have been concentrated to various phenomena occurring during Mach reflection of solitons as well as during crossing of long-crested solitonic waves in shallow water (e.g., [10-13]). Several deeply interested results have been obtained, and it's a pity that there exists no comprehensive overview of the material.

Somewhat surprisingly, this mechanism was applied as an explanation of freak waves in shallow water (Figure 1) only few years ago [5]. The mechanism in question was actually proposed to explain unusually high ship wakes occurring in areas with heavy fast ferry traffic. Such a delay is even more curious because of the wide knowledge of the existence of long solitonic waves in shallow areas and damage caused by them already on the textbook level [14].

Although it is attractive to use the results obtained, for example, in [8] for Boussinesq solitons, for characterising the features of the KP solitons, doing so is, strictly speaking, generally incorrect, because the governing equations and their solutions are different. The relevant analysis is not very difficult and in many cases straightforward, for the analytical expressions of the two-soliton solution of the KP equation are reasonable. Yet these expressions describe quite a complex family of surfaces depending on four parameters of the interacting waves – their amplitudes and propagation directions.

Freak features of soliton interactions in the KP framework: height, steepness, water level setup and potentially frequent occurrence

Several features of interactions of KdV solitons – interpreted as properties of two-soliton solutions of the KP equation – are exactly the ones usually attributed to freak waves. As mentioned above, in certain cases, such nonlinear coupling produces a particularly high wave hump. It is fairly easy to show that interactions of equal amplitude solitons may lead to water surface elevation up to four times as high as the amplitude of the counterparts [5].

For solitons of arbitrary amplitude (Figure 2) the maximum elevation, described in terms of wave numbers k_1 and k_2 of the counterparts, is

$$\eta_{\max} = \frac{(k_1 + k_2)^2}{2} - \frac{2k_1 k_2}{A_{12}^{1/2} + 1}, \tag{1}$$

where the amplitudes (maximum elevations) of the counterparts are $a_1 = k_1^2/2$, $a_1 = k_1^2/2$ and A_{12} is a certain combination of incoming wave amplitudes and propagation directions, and is called the phase shift parameter ($A_{12} > 1$ in cases relevant to freak wave generation). This result can be obtained in a straightforward way under the assumption that the global maximum of the relevant surface occurs at the 'center' of interactions of the solitons [15]. This assumption can be easily verified for equal amplitude solitons but the relevant proof in general case is nontrivial [16]. The maximum of amplitude occurs in the limiting process $A_{12} \to \infty$ when the length of the common crest of the solitons infinitely increases and the incoming solitons actually merge into one (resonant) soliton.

Another important feature of soliton interactions, which is perhaps even more relevant to freak wave business, is that the slope of the water surface at the wave front may encounter eightfold increase. This was first demonstrated as an asymptotic feature of interactions of equal amplitude solitons in [17] occurring somewhere infinitely far from the classical viewpoint of soliton interactions – the bifurcation point of the crests. The result was extended to the case of solitons of arbitrary amplitudes in [18] where also relevant exact expressions were derived.

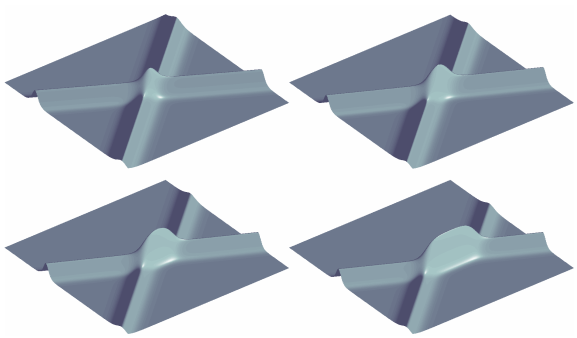

Figure 2. Surface elevation owing to interaction of unequal amplitude solitons with different angles between their crests.

In fact, such extreme slope amplification is not unexpected in case when the structure resulting from nonlinear interactions is a perfect KdV soliton. The profile of a single KdV soliton is $\eta = \frac{1}{2}k^2 \cosh^{-2} \frac{1}{2}kx$. The maximum slope of its front is $S_{\max} = k^3/3\sqrt{3}$. If in the limiting case $A_{12} \to \infty$ the composite structure (or a part of it) is a KdV soliton, it has to be a so-called resonant soliton, which concentrates energy of the incoming solitons into one structure. If the wave numbers of the original counterparts are k_1 and k_2, the amplitude and wave number of the resonant soliton are $\frac{1}{2}(k_1 + k_2)^2$ and $k_\infty = k_1 + k_2$, respectively. The maximum slope of the resonant soliton is therefore

$$S_{\max}^{\infty} = \frac{k_\infty^3}{3\sqrt{3}} = \frac{(k_1 + k_2)^3}{3\sqrt{3}}. \tag{2}$$

For equal amplitude solitons, the slope of the resulting structure is $2^3 = 8$ times as steep as the slopes of the incoming solitons. The drastic amplification of the slope of the water surface thus reflects the well-know fact that the higher a KdV soliton is, the slimmer is its profile. The resonant KdV soliton is simply higher and therefore narrower than the incoming solitons.

This simple reasoning is supported by more sound analysis in [19]. Heuristically, it is consistent that the resonant structure belongs to the same class of solutions as the interacting counterparts. This hypothesis has been partially confirmed by proving that a specific asymptotic limit of the resulting structure at the exact resonance case satisfies the same equation as the incoming solitons. Although this is an important step towards establishing the nature of the resonant structure, strictly speaking, this result does not prove that the resonant soliton is a soliton of the same type as the interacting solitons. The proof that in the limiting case of exact resonance the central part of the interaction pattern tends to a new KdV soliton is given in [19].

A couple of good news follow from the analysis of the two-soliton solutions. First, the point of 'bifurcation' of the crests of the incoming solitons and the composite structure always lies outside of the area when surface elevation twice or more exceeds the amplitudes of the incoming solitons [17]. This feature might console the possible observer of the exceptionally high wave hump occurring in the framework of the KP equation. If there is a particularly strong hit, then it comes alone.

Second, the extent of the area where considerable amplification of wave heights may take place is moderate. Substantial areas of extreme surface elevation may only occur if the heights of the incoming waves, their intersection angle and the local water depth are specifically balanced. The diameter of the relevant area is proportional to $\ln A_{12}$ as $A_{12} \to \infty$ [16,17]. Thus, the fraction of sea surface occupied by extreme elevations apparently is small as compared with the area where solitonic waves potentially may interact.

A bad news, first recognised by Prof. E. Pelinovsky (private communication, 2003, see also [1]) is that the extremely high parts of the composite structures are a sort of possibly long-living waves. The interaction pattern is a two-dimensional structure generally of limited length but moving over a sea area. Theoretically, it has unlimited life-time and may can cross large sea areas in favourable conditions. Thus, one should also account for the expected life-time of the nonlinear wave humps (additionally to the fraction of the sea area covered by extreme elevation at a certain time instant) when estimating the probability of occurrence of near-resonant interaction solitons. This feature may potentially contribute to the renowned feature of freak waves that they occur much more frequently than it might be expected from surface wave statistics.

Another, perhaps less known feature of freak waves is that some of them (for example, the Draupner wave) are accompanied by a short-time increase in the mean water level whereas during the passage of groups of ‘non-freak’ large waves a certain water level dropdown occurs [20]. The effect of nonlinear interactions in the framework of the KP equation that substantially modifies the profile of the two-soliton solution and simultaneously brings into motion a large amount of water above the mean sea level may be a part of the new physics of freak waves for relatively shallow areas.

Dynamical implications

The extraordinary steepness of the front of the composite structure (and, as a result, the front of the moving surface corresponding to a two-soliton solution) is a most interesting feature of the soliton interactions in this framework. This nice feature may be the main reason why the wave hump near the interaction centre frequently breaks until it reaches its theoretically maximum height. Even if stabilised by the interplay of dispersion and nonlinearity, thin water 'tongue' may easily become unstable. The sea bottom is never perfect, and the moving interaction pattern may meet conditions where KP equation might be invalid or where the two-soliton solution does not exist. In such situations it may breaks as any other water wave does. The possibility of breaking of the high and nonlinear wave hump makes a hit by a near-resonant pattern exceptionally dangerous [5].

In populated or industrial areas, a possible hit of the central part of a near-resonant long-crested high wave hump on an entrance of a channel (harbour entrance, river mouth etc.) may cause serious consequences. The reason is that this structure is basically different from a superposition of two linear wave trains at the same place. Linear waves continue to move in their original directions, and result in a system of interfering waves in the channel. If a near-resonant wave hump enters a channel, it concentrates energy of both incoming solitons in one structure, the further behaviour and stability of which is yet unclear [5].

The discussed mechanism is only applicable in relatively shallow water. In order to become effective, there should exist at least two long wave systems propagating under certain angle, and the whole situation should satisfy the restrictions of the KP equation. Such situation may happen when two swell systems propagate into a certain shallow area. More frequently, such conditions may occur at downwind side of a small island or shallow area where topographically refracted waves meet each other in a formally sheltered region.

The potential applicability of the discussed mechanism is extremely important for shipping safety, because marine transport is only reasonable when ships at times make a port call, that is, cross the shallow and/or specifically sheltered area near the ports.

References

1. Kharif, C., Pelinovsky, E., Physical mechanisms of the rogue wave phenomenon, *Eur. J. Mech. B Fluids* **22**, 603–634, 2003.
2. Kadomtsev, B.B., Petviashvili, V.I., The stability of solitary waves in weakly dispersive media, *Dokl. Akad. Nauk SSSR* **192**, 539–541, 1970.
3. Zakharov V.E., Shabat, A.B., A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem I, *Funct. Anal. Appl*. **8**, 226–235, 1974.
4. Satsuma, J., N-soliton solution of the two-dimensional Korteweg-de Vries equation, *J. Phys. Soc. Japan* **40**, 286–290, 1976.
5. Peterson, P., Soomere, T., Engelbrecht, J., van Groesen, E., Interaction soliton as a possible model for extreme waves in shallow water, *Nonlinear Processes in Geophysics* **10**, 503-510, 2003.
6. Newell, A.C., Redekopp, L.C., Breakdown of Zakharov-Shabat theory and soliton creation, *Phys. Rev. Lett*. **38**, 377–380, 1977.
7. Perroud, P.H., The solitary wave reflection along a straight vertical wall at oblique incidence, University of California, Berkeley, IER Rept. 99-3 PH, 1957.
8. Miles, J. W., Resonantly interacting solitary waves, *J. Fluid Mech*. **79**, 171–179, 1977.
9. Yue, D.K.P., Mei, C.C., Forward diffraction of Stokes waves by a thin wedge, *J. Fluid Mech*. **99**, 33–52, 1980.
10. Freeman, N.C., Soliton interactions in two dimensions, *Adv. Appl. Mech*. **20**, 1–37, 1980.
11. Hammack, J., Scheffner, N., Segur, H., Two-dimensional periodic waves in shallow water, *J. Fluid Mech*. **209**, 567–589, 1989.
12. Hammack, J., McCallister, D., Scheffner, N., Segur, H., Two-dimensional periodic waves in shallow water. Part 2. Asymmetric waves, *J. Fluid Mech*. 2**85**, 95–122, 1995.
13. Peterson, P., van Groesen, E., A direct and inverse problem for wave crests modelled by interactions of two solitons, *Physica D* **141**, 316–332, 2000.

14. Dean, R.G, Dalrymple, R.A., Coastal processes with engineering applications, Cambridge University Press, 2002, Chapter 4.
15. Duan, W.-S., Shi, Y.-R., Hong, X.-R., Theoretical study of resonance of the Kadomtsev-Petviashili equation, *Phys. Lett. A* **323**, 89–94, 2004.
16. Soomere, T., Interaction of Kadomtsev-Petviashvili solitons with unequal amplitudes, *Phys. Lett. A* **332**, 74–81, 2004.
17. Soomere, T., Engelbrecht, J., Extreme elevations and slopes of interacting solitons in shallow water, *Wave Motion*, **41**, 179–192, 2005.
18. Soomere, T., Engelbrecht, J., Extreme elevations and slopes of interacting Kadomtsev-Petviashvili solitons in shallow water, submitted to *Proc. Int. Conf. Rogue Waves 2004, SeaTechWeek 2004, Le Quartz, Brest, France, 20-22 October 2004*, IFREMER, Brest 2004.
19. Soomere, T., Engelbrecht, J., Weakly two-dimensional interaction of solitons in shallow water, submitted to *European J. of Mech. B Fluids* 2006.
20. Walker, D.A.G., Taylor, P.H., Taylor, R.E., The shape of large surface waves on the open sea and the Draupner New Year wave, *Applied Ocean Research* **26**, 73–83, 2004.