ROGUE WAVES ON CURRENTS WITH VERTICAL SHEAR

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Motivations

- There is a vast anecdotal evidence suggesting that rogue waves are much more frequent in areas characterised by the presence of strong currents. Perhaps, the best known are those associated with ill famed Aguilhas current, off the coast of South-East Africa. For the Aguilhas current, for example, the practical wisdom accumulated in the sailing directions reads: ***avoid the Aguilhas current***.

- There was no lack of attempts to explain the high probability of rogue waves on currents. In all the papers I am aware of, only the ***horizontal*** non-uniformity of the current was taken into account.

- However, so far no studies have targeted the specific nonlinear mechanisms responsible for the rogue wave formation on strong currents and, in particular, the possible role of ***vertical shear***. The present paper is the first aimed at this gap.

Goal

To investigate plausible nonlinear physical mechanisms of rogue wave formation on strong currents due to vertical shear.

In the absence of shear currents only very few possible mechanisms have been explored so far:

- ***Spatio-temporal focusing***
  This mechanism is well understood (*Pelinovsky* and his group) and its probability could be calculated.
- ***Modulational (Benjamin-Feir) instability***
  Sufficiently narrow-banded wave fields are known to be modulationally unstable. Ample numerical evidence supports the view that the further evolution of BF instability might lead to anomalously high waves.

  There is also some experimental evidence suggesting a link between the probability of freak waves and the BF instability (*Janssen*).

  The issue was extensively studied theoretically and numerically (*Peregrine, Osborne, Onorato, Janssen, Zakharov*)

The main idea

What are the main consequences of adding a vertical shear into the picture?

First, the standard four wave interactions are slightly modified and, the characteristics of modulational instability are also accordingly modified.

Second, a shear current supports its own modes of motion which can interact with water waves in the lower order (quadratic and not cubic) of nonlinearity. These current modes could be induced by water waves and have a noticeable feedback effect on water wave dynamics. It is this mechanism we investigate from the rogue wave perspective.

Assumptions

- *ideal fluid*;
- *unidirectional current*;
- *waves' weak nonlinearity:* $\epsilon \ll 1$;
- *moderate weakness of the current:*
  $\mu = U_s/C \ll 1$
- *weak nonlinearity of the induced motions in water:* $\epsilon_L = O(u_L/U_s) \ll 1$
- *separation of scales*:
  $L_x \gg L_y \gg \lambda$
  $k_x L_x \gg 1, k_y L_y \gg 1$

Employing the small parameters we first derive the simplest model of coupled evolution of a narrow-banded wave packet, characterised by:

an envelope amplitude A,

and

wave induced motions $\mathbf{u}_L$

Model

The basic equations

$$\begin{aligned} D_t u + wU' + P_x &= f^{(1)} = & -(\boldsymbol{u}\boldsymbol{\nabla})u \\ D_t v + P_y &= f^{(2)} = & -(\boldsymbol{u}\boldsymbol{\nabla})v \\ D_t w + P_z &= f^{(3)} = & -(\boldsymbol{u}\boldsymbol{\nabla})w \\ u_x + v_y + w_z &= 0 \end{aligned} \tag{1}$$

where

$$\boldsymbol{U} = \{U(z), 0, 0\}, \qquad D \equiv \partial_t + U\partial_x, \qquad U' \equiv \partial_z U \tag{2}$$

with the standard boundary conditions at the free surface $z = \eta(x, y, t)$

$$\eta_t + ([\boldsymbol{U} + \boldsymbol{u}]\boldsymbol{\nabla})\eta = w, \qquad P = 0, \tag{3}$$

and

$$w = 0 \qquad \text{at the bottom or at infinity}$$

The asymptotic expansion

Consider evolution of a narrow packet of free water waves on a shear current. We look for the solution in the form of asymptotic expansion

$$\begin{aligned} \boldsymbol{u} &= \epsilon[A(\epsilon_x x, \epsilon_y y, \epsilon_t t)\mathrm{e}^{i\Theta} + c.c.] + \epsilon^2 \boldsymbol{u}_2 + ... \\ \boldsymbol{u}_L &= \mu \boldsymbol{U}(z) + \epsilon_L \boldsymbol{u}(z, \epsilon_x x, \epsilon_y y, \epsilon_t t) + ... \end{aligned} \tag{4}$$

where subscript $(...)_L$ indicates induced low-frequency packet-scale motions, parameter ϵ_L specifies their smallness.

$$\epsilon = \frac{u}{C}, \qquad \textit{the nonlinearity parameter of the surface waves}$$

u is horizontal velocity of the particles in the waves, C is the phase velocity of the surface wave.

$$\mu = \frac{U_{\max}}{C}.$$

Typical values for the ocean are $\epsilon \simeq 10^{-2} - 10^{-1}$, $\mu \simeq 10^{-2} - 10^{-1}$.

The scaling

The scaling is *a priori* not obvious. It is

$$\epsilon_L \sim \epsilon_y^2 \sim \epsilon\sqrt{\mu}, \qquad \epsilon_t \sim \epsilon\sqrt{\mu}, \tag{5}$$

while

$$\epsilon_x \ll \epsilon_y^2, \qquad \epsilon_L \ll \mu, \qquad \epsilon \ll \sqrt{\mu}.$$

If we, or simplicity, assume $\mu \sim \epsilon$, scaling (5) simplifies to

$$\epsilon_L \sim \epsilon^{3/2} \qquad \epsilon_y \sim \epsilon^{3/4}, \qquad \epsilon_t \sim \epsilon^{3/2}. \tag{6}$$

The key element here is the assumption of very slow $x-$ dependence

$$\epsilon_x \ll \epsilon_y^2.$$

The faster $x-$ dependence due to other mechanisms is "filtrated". The adopted scaling implies that we consider the regimes where the packet transverse diffusion prevails over longitudinal transport.

At the order $O(\epsilon\epsilon_y^2)$, equating the terms proportional to $\exp[i\Theta]$, we arrive at the parabolic equation

$$iA_T + \frac{c_g}{2k}A_{YY} + A\tilde{u}_L = 0 \tag{7}$$

where the last term is the correction to the group velocity owing to the Doppler shift due to the ***nonsteady*** part of the shear current u_L, and $\tilde{u}_L$ is the vertically integrated x-component of $\boldsymbol{u}_L$ with a weighting function

$$\tilde{u}_L = 4k\int_0^\infty u_L(z,\zeta,Y,T)\exp(-2kz)\mathrm{d}z. \tag{8}$$

To close the system we have to express $\tilde{u}_L$ in terms of the complex envelope amplitude $A(Y,T)$.

The full linearised vorticity equation in the $(\boldsymbol{x},t)$-space for the vertical component of the induced motion w_L reads (we drop the subscript $\{\}_L$)

$$(\partial_T + U\partial_X)(w'' + \Delta w) - U'' w_X = -(\frac{1}{2C})\chi(z)\partial^2_{YY} \mid A(X,Y,T) \mid^2;$$
$$\chi(z) = (U^{'''} - 4kU'' + 8k^2U')\mathrm{e}^{-2kz}. \qquad (9)$$

Note, that to calculate the RHS we have to take into account non-potential corrections due to shear to water waves.

In the leading order we find

$$\partial_T w'' = -\chi(z)\left(\frac{1}{2C}\right)\partial^2_{YY} \mid A(Y,T) \mid^2 \tag{10}$$

Separating the variables

$$w_L = B(Y,T)W(z) \tag{11}$$

we find $W(z)$

$$W'' = \chi(z); \tag{12}$$

$W(z) = 0$ at the surface $z = 0$ and have to tend to a constant at infinity.

$$W(z) = \int_0^z \int_0^{z_2} \chi(z_1)\mathrm{d}z_1\mathrm{d}z_2 - z\int_0^\infty \chi(z_1)\mathrm{d}z_1. \tag{13}$$

To the leading the time dependence is given by the equation

$$\partial_t B = -\left(\frac{1}{2C}\right)\partial^2_{YY} \mid A \mid^2 . \tag{14}$$

To express u_L in terms of the previously found w_L, we use the x-projection of the Euler equations

$$\partial_t u_L + U' w_L = o({\varepsilon_L}^2)$$

and taking into account the derived explicit expressions for w_L), we find

$$\partial_t u_L - U' W(z) \left(\frac{1}{2C}\right) \partial_t^{-1} \partial_{YY} \mid A \mid^2 = 0.$$

Thus, the explicit relation between u_L and A now reads

$$\partial^2_{tt} u_L = \left(\frac{1}{2C}\right) U' W(z) \partial^2_{YY} (\mid A \mid^2).$$

On integrating u_L with the known weighting function we finally express $\tilde{u}_L$ in terms of A

$$\partial_{tt}^2 \tilde{u}_L = r_1 \partial_{YY}^2 (\mid A \mid^2); \tag{15}$$

where

$$r_1 \equiv \frac{2k}{C}[\int_0^\infty U'(z) W(z) \exp(-2kz) \mathrm{d}z]$$

and $W(z)$ is given by

$$W(z) = \int_0^z \int_0^{z_2} \chi(z_1) \mathrm{d}z_1 \mathrm{d}z_2 - z \int_0^\infty \chi(z_1) \mathrm{d}z_1.$$

$$\chi(z) = (U''' - 4kU'' + 8k^2 U') \mathrm{e}^{-2kz}$$

Thus we've closed the system.

The final form In the appropriately re-scaled variables

$$
\begin{aligned}
iA_t + A_{yy} + AQ &= 0 \\
\partial^2_{tt} Q &= s_1 \partial^2_{yy} (\mid A \mid^2);
\end{aligned}
\qquad (16)
$$

The only coefficient $s_1 = \pm 1$ is determined by the sign of r_1

For the simplest model profiles $U(z)$ $s_1 = 1$ for waves propagating upstream, while for waves propagating downstream $s_1 = -1$.

The system is a 1-D version of the Zakharov's eq-s used in plasma physics to describe Langmuir turbulence ($\partial^2_{tt} Q - \partial^2_{yy} Q = \partial^2_{yy} (\mid A \mid^2)$).

Interpretation: Self-induced Wave Guides

Similarly to plasma physics context, the system describes interaction of a wave packet with a self-induced waveguide.

True and "dark" NLS solitons describe drifting streaks

$$A = a/\cosh\left[\frac{a}{V}(y - Vt)\right]$$

which become more and more narrow as V tends to zero.

Stability

For the perturbations of the form $\exp[-i\Omega t + i\kappa y]$ the dispersion relation is

$$\Omega^2(\kappa) = \frac{\kappa^4}{2}\left[1 \pm \sqrt{1 - \frac{8sa_0^2}{\kappa^4}}\,\right] \qquad (17)$$

a_0 is the amplitude of the unperturbed wave.

The upstream propagating waves ($s = 1$) have a high wavenumber cut-off:

$$\kappa_{cutoff} = 2^{3/4}\sqrt{a_0} \sim \mu^{1/4}\sqrt{\epsilon} \sim \epsilon^{3/4}$$

This justifies *a posteriori* our key assumption that the packet is narrow or that $\epsilon_y \ll 1$.

The maximal growth rate for upstream waves $\Gamma_{max} = (Im\Omega)_{max} \sim \sqrt{\mu}\epsilon \sim \epsilon^{3/2}$

Brief conclusions and long discussion

- In the presence of strong vertical shear the waves' nonlinear interaction due to shear current (the induced scatterring) could be the strongest. The transverse instability timescale is $\sim \sqrt{\mu}\epsilon \sim \epsilon^{3/2}$.

- The derived simplest model is adequate for description of upstream propagating waves interacting with the self-induced waveguides due to the existence of cut-off at a relatively small κ: $\mu^{1/4}\epsilon^{1/2} \sim \epsilon^{3/4}$.

- Strictly speaking, the model is inadequate for the case of waves propagating downstream.

- The emergence of self-induced waveguides and their possible interaction could lead to strong transverse intermittency of the wave field. It could be further enhanced by interaction of the waveguides since they move cross-wise.

There is an open question: are the waveguides stable and longlived?

Collapse (?)

Let us represent the envelope complex amplitude A in the system as follows

$$A = \Phi(y,t)\exp[i\int \sigma^2(t)dt],$$

which results in the equivalent form of the system

$$\begin{aligned} i\Phi_t - \sigma^2(t)\Phi + \Phi_{yy} + Q\Phi = 0 \\ Q_{tt} = s(\Phi^2)_{yy} \end{aligned} \tag{18}$$

Consider solutions to this system in the form

$$\Phi = \frac{1}{t-t*}\Psi(\frac{y}{(t-t*)^2})$$

$$Q = \frac{1}{(t-t*)^4}q(\frac{y}{(t-t*)^2}) \tag{19}$$

$$\sigma^2 = \frac{\sigma_0^2}{(t-t*)^4}$$

Here t^* stands for the moment of the singularity formation. Introducing new independent variables

$$\tau \equiv (t - t*), \qquad \zeta = \frac{y}{(t-t*)^2} \tag{20}$$

we insert the ansatz into the system retaining the leading order terms in τ^{-1}.

After some algebra we arrive at a relatively simple ODE system

$$-\sigma_0^2\Psi + \Psi'' - \Psi q = 0$$

$$4q''\zeta^2 = s(\Psi^2)''$$

for which some solutions could be found in terms of power series for large ζ.

The solutions describe a possible scenario of how the field might develop the space-time singularity, but I don't have a theorem specifying the class of initial conditions which would lead to blow up.

Obvious generalization: from a single packet to a continuum

$$i\partial_T A_k + \frac{c_g}{2k}\partial^2_{YY}A_k - kA_k\tilde{u}_L = 0;$$
$$\partial^2_{TT}\tilde{u}_L = \partial^2_{YY}(\textstyle\int r_1(k) \mid A_k \mid^2 \mathrm{d}k); \tag{21}$$

$A_k(y,t)$ is an *amplitude* of Fourier component of the wave field, the subscript designates the streamwise wavenumber (the Fourier transform is done over the x-coordinate only).

The system describes coupled evolution of the *angular* wave field and self-induced waveguides.

The Final Conclusion

To say anything specific about the role of the proposed physical mechanism in any concrete location a further study is needed.