Anomalous probability of strong waves

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Plan of the talk:

- Statistical waves – “Wave Turbulence”.
- Non-gaussianity of wave PDF; intermittency.
- Discreteness effects.

What is Wave Turbulence?

WT describes a stochastic field of weakly interacting dispersive waves.

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How can we describe WT?

- Hamilitonian equations for the wave field.
- Weak nonlinearity expansion. Separation of the linear and nonlinear timescales.
- Statistical averaging, - closure.

Free surface motion

r is a 2D vector in horizonal plane; z is the vertical coordinate

$\eta(r)$ *surface elevation*

$v(r, z) = \nabla\phi$ *velocity*

$H = T + U,$ *total energy*

$$T = \frac{1}{2}\int d^2 r \int_{-\infty}^{\eta} (\nabla\phi)^2 dz, \qquad U = \frac{1}{2} g \int \eta^2 d^2 r,$$

where g is the acceleration of gravity.

Hamiltonian system *(Zakharov 1968)*

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \quad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}$$

Zakharov equation

$$i\dot{b}_l = \epsilon \sum_{\alpha\mu\nu} W^{l\alpha}_{\mu\nu} \bar{b}_\alpha b_\mu b_\nu e^{i\omega^{l\alpha}_{\mu\nu} t} \delta^{l\alpha}_{\mu\nu}$$

where b_l is the wave action variable in the interaction representation, $l \in \mathcal{Z}^2$, $W^{l\alpha}_{\mu\nu} \sim 1$ is an interaction coefficient, $\omega^{l\alpha}_{\mu\nu} = \omega_l + \omega_\alpha - \omega_\mu - \omega_\nu$, and ϵ is a small nonlinearity parameter.

Deep water waves, $\omega^2 = gk$, W is complicated (Krasitski 1992)

Statistical objects in WT

- Amplitude & phase: $a_k = A_k \Psi_k;\ \Psi_k = \exp(i\varphi_k)$.
- Spectrum $n_k = \langle A_k^2 \rangle$ e.g. Kolmogorov-Zakharov spectrum
- N-mode PDF: probability
  for A_k^2 to be in $[s_k,\ s_k + ds_k]$ and
  for Ψ_k to be in $[\xi_k,\ \xi_k + d\xi_k]$,

$$P^{(N)}\{s,\xi\} = \langle \delta(s-A^2)\, \delta(\xi-\Psi) \rangle;$$

$s=\{s_1, s_2, \ldots, s_N\};\ A=\{A_1, A_2, \ldots, A_N\};$
$\xi=\{\xi_1, \xi_2, \ldots, \xi_N\};\ \Psi=\{\Psi_1, \Psi_2, \ldots, \Psi_N\}.$

Random Phase & Amplitude (RPA) wavefield:

- All the amplitudes and the phase factors are independent random variables,

$$\mathcal{P}^{(N)}\{s,\xi\} = \prod_{l \in \mathcal{B}_N} P_l^{(a)}(s_l) P_l^{(\psi)}(\xi_l)$$

- The phase factors are uniformly distributed on the unit circle in the complex plane.

$$P_l^{(\psi)}(\xi_l) = 1/2\pi$$

RPA fields are not Gaussian.

- Gaussian distribution means

$$P^{(a)}(s) \sim e^{-s/n}.$$

- RPA does not fix the amplitude PDF.

Weak nonlinearity expansion

Choose T in between the linear and nonlinear timescales:

$$2\pi/\omega \ll T \ll 1/\omega\epsilon^2.$$

Find by recursive substitution:

$$a_l(T) = a_l^{(0)} + \epsilon a_l^{(1)} + \epsilon^2 a_l^{(2)}$$

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Evolution of WT statistics

- Substitute value of $a_k(T)$ into the PDF definition.
- Apply RPA to $a_k(0)$.
- Replace $[P(T)-P(0)]/T$ with $\partial_t P$.

Equation for the N-mode PDF
(Choi, Lvov & SN, 2004)

$$\dot{\mathcal{P}} = -\int \frac{\delta F_j}{\delta s_j}\, dk_j,$$

Where F_j is the j-component of the flux,

$$F_j = 4\pi\epsilon^2 \int_{123} W^{j1}_{23}\delta(\tilde{\omega}^{ji}_{23})\delta^{j1}_{23} s_1 s_2 s_3 s_j \left(\frac{\delta}{\delta s_j} + \frac{\delta}{\delta s_1} - \frac{\delta}{\delta s_2} - \frac{\delta}{\delta s_3}\right) P,$$

$$\text{with } \tilde{\omega}^{l\alpha}_{\mu\nu} = \omega^{l\alpha}_{\mu\nu} + \Omega_l + \Omega_\alpha - \Omega_\mu - \Omega_\nu$$

Use of the N-mode PDF

- Validation that RPA holds over the nonlinear evolution time
- Non-Gaussian statistics of the wave amplitudes

Single-mode staitstics

Eqn. for the 1-mode PDF (Choi, Lvov, SN, 2003):

$$\dot{P} + \partial_s F = 0,$$

$$\text{where } F = -s(\gamma P + \eta \partial_s P)$$

$$\gamma_k = 4\pi\epsilon^2 \sum_{234} \tilde{W}^{k4}_{23}(-n_2 n_3 + n_3 n_4 + n_2 n_4)$$

$$\eta_k = 4\pi\epsilon^2 \sum_{234} \tilde{W}^{k4}_{23} n_2 n_3 n_4$$

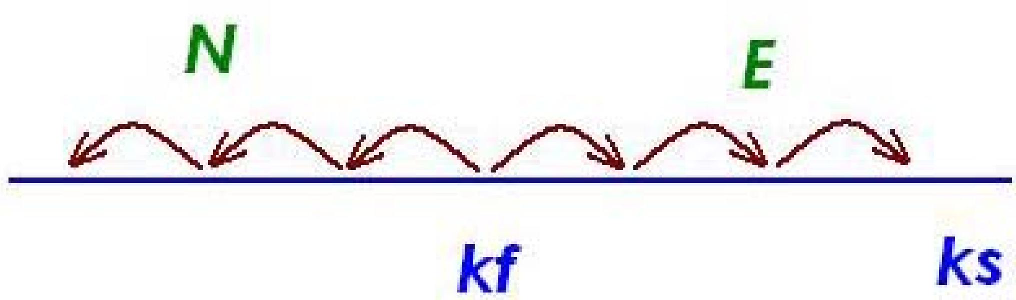
Kinetic equation for spectrum

Taking 1st moment of the 1-mode PDF eqn:

$$\begin{aligned}\dot{n} &= -\gamma n + \eta \\ &= 4\pi\epsilon^2 \sum_{234} \tilde{W}_{23}^{k4} n_2 n_3 n_4 n_k \left(\frac{1}{n_k} + \frac{1}{n_4} - \frac{1}{n_2} - \frac{1}{n_3} \right)\end{aligned}$$

Hasselmann, 1963

Kolmogorov-Zakharov spectra



- Power-law spectra describing a down-scale energy cascade and an up-scale wave-action cascade

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Steady state PDF

Choi, Lvov, SN, 2003

$$-s(\gamma P + \eta \partial_s P) = F = \text{const.}$$

$$P = P_{hom} + P_{part} = \text{const} \exp\left(-s/n\right) - F/(s\gamma)$$

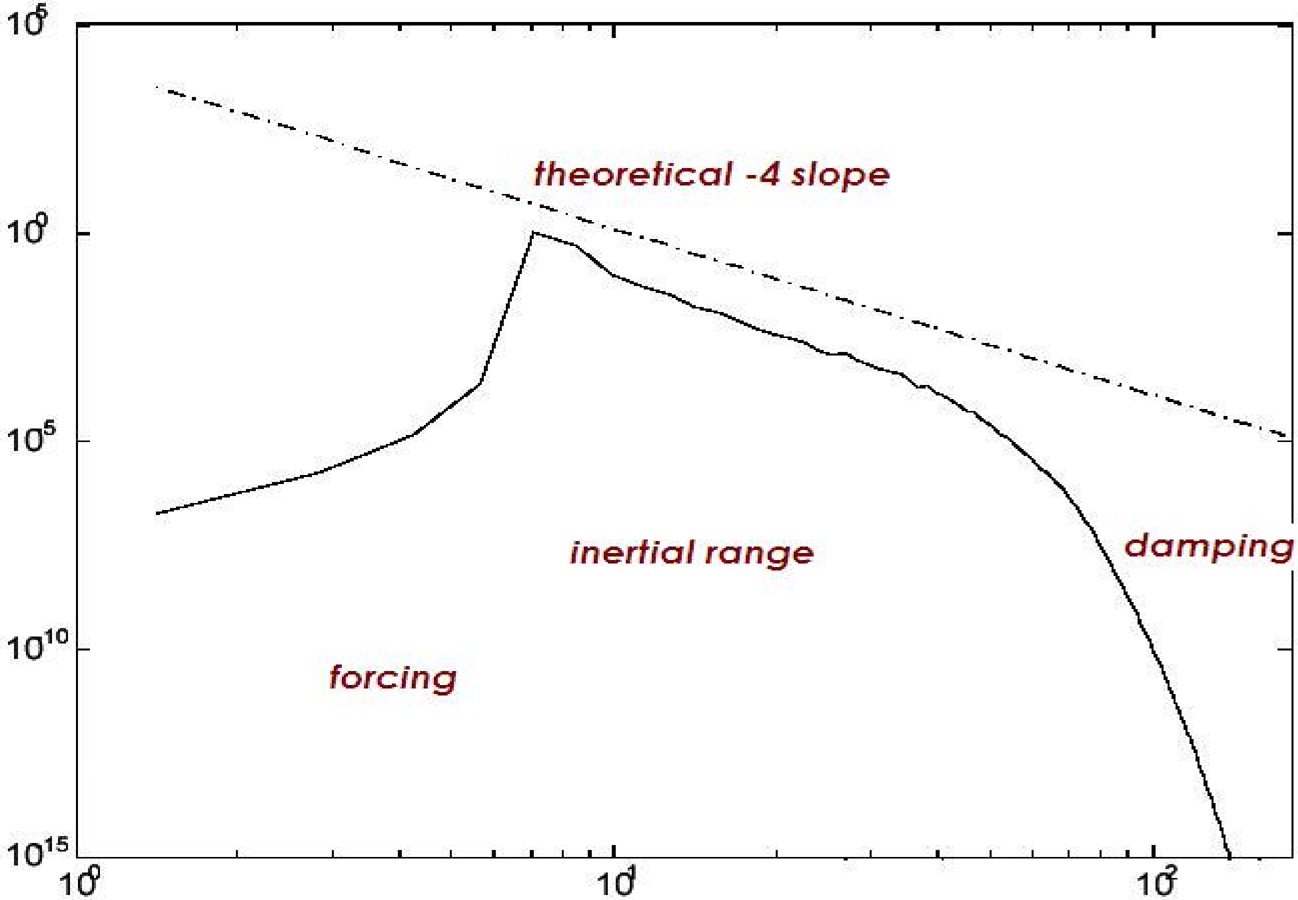
$$s \gg n_k ,$$

- Gaussian core, non-gaussian tail

Direct Numerical Simulations

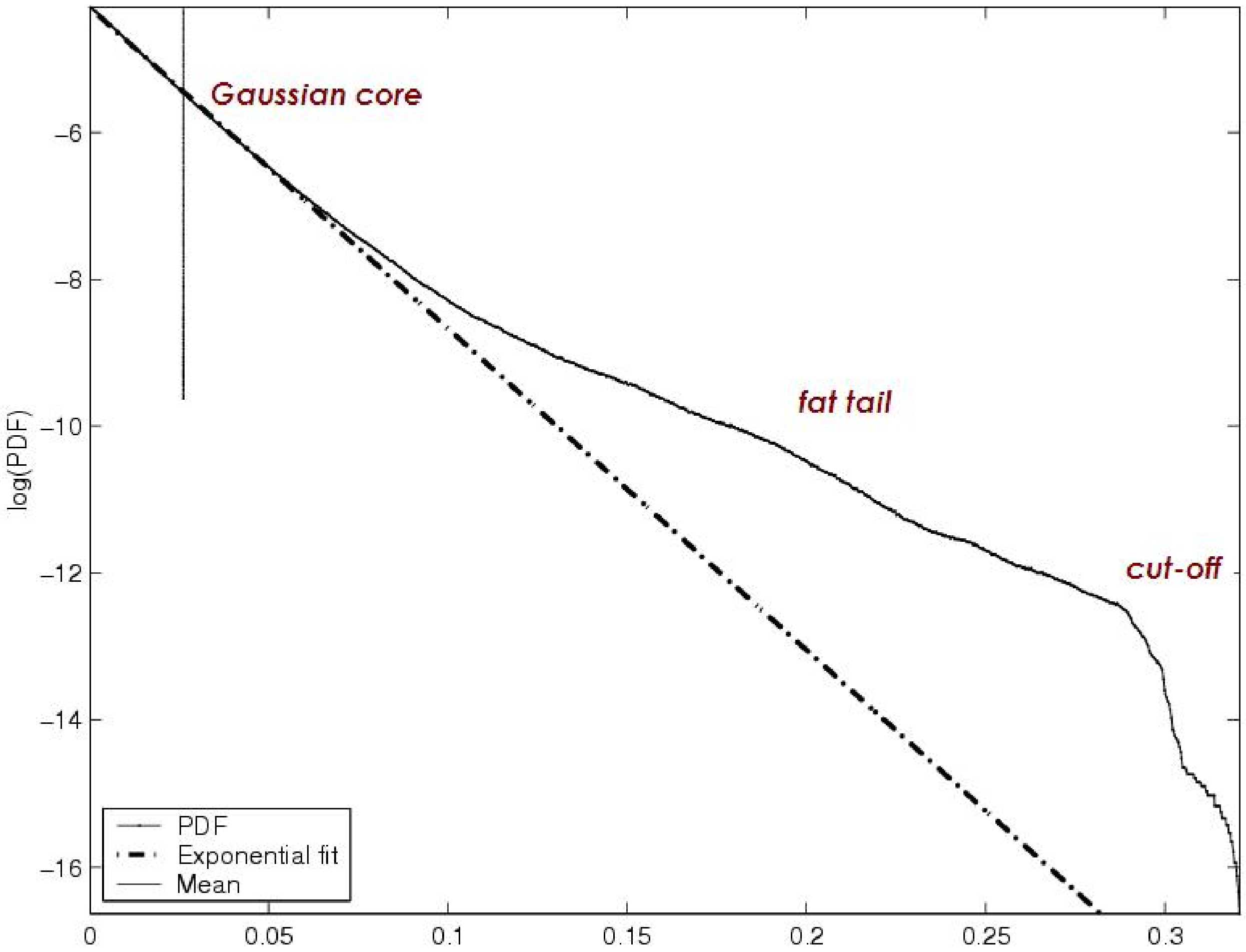
- Truncated (at the 3rd order in amplitude) Euler equations for the free water surface.
- Pseudo-spectral method 256X256.

Energy spectrum



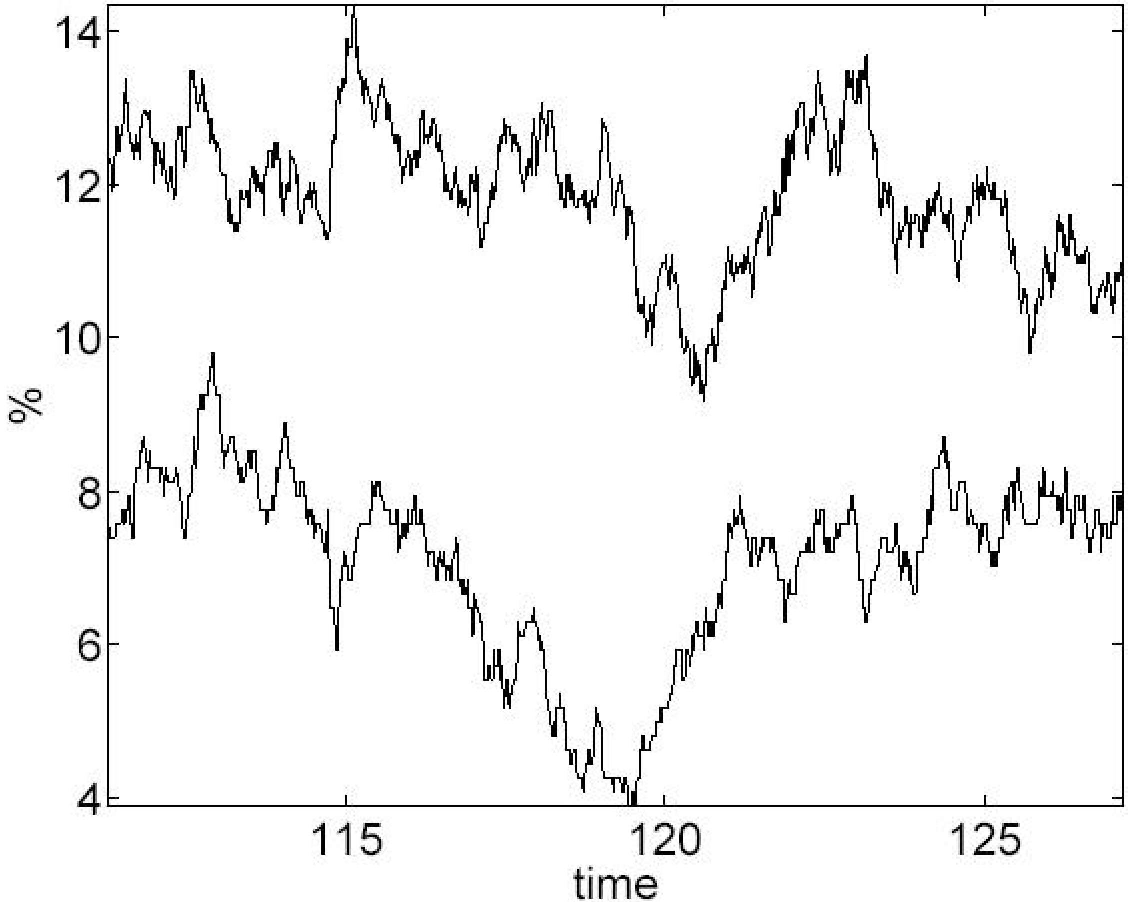
Onorato et al' 02, Dyachenko et al'03, Nakoyama'04, Lvov et al'05

One-mode PDF



- Anomalously high amplitude of large waves.

“Sandpile avalanches” in discrete turbulence



- Nonlinear activity at k_1 and k_2 are correlated ($k_2 > k_1$),
- It is time delayed and amplified at k_2 with respect k_1.
- “sandpile avalanches”.

Cycle of discrete turbulence

- Weak turbulence at forcing scale – no resonance, no cascade.
- Energy accumulation, growth of nonlinearity.
- Nonlinear resonance broadening, cascade activation – “avalanche”.
- Avalanche drains energy from the forcing scale, -> beginning of the cycle.

Summary

- Generalised WT description: PDF.
- Anomalous distribution of waves with high amplitudes
- Discreteness effects: exact and quasi-resonances, sanpile behavior of cascade.

What follows is an extra material that might be useful if questions asked

Phases vs Phase factors

- Illustration through an example :

$$\phi_{1,2} = 2\pi N + r_{1,2} \qquad \psi_{1,2} = e^{i\phi_{1,2}}$$

  - Phases ϕ_1 and ϕ_2 are correlated, because

$$\langle \phi_1 \phi_2 \rangle - \langle \phi_1 \rangle \langle \phi_2 \rangle = 4\pi^2 (\langle N^2 \rangle - \langle N \rangle^2) > 0$$

  - Phase factors ψ_1 and ψ_2 are statistically independent, $\langle \psi_1 \psi_2 \rangle - \langle \psi_1 \rangle \langle \psi_2 \rangle = 0$

Mean phase

- Expression for phase $\phi_j = \Im \ln a_j$
- Evolution eqn.

$$\langle \dot{\phi_j} \rangle = 4\epsilon^2 \int \left[|V^j_{mn}|^2 \mathcal{P}\left(\frac{1}{\omega^j_{mn}}\right) \delta^j_{m+n} - |V^m_{jn}|^2 \mathcal{P}\left(\frac{1}{\omega^m_{jn}}\right) \delta^m_{j+n} (n_m - n_n) \right] n_j \, dk_m dk_n$$

the mean value of the phase is steadily changing over the nonlinear time and it would be incorrect to assume that phases remains uniformly distributed in [-π,π].

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Dispersion of the phase

$$\sigma_k = \langle \phi_k^2 \rangle - \langle \phi_k \rangle^2$$

$$\dot{\sigma}_k = \eta_k / n_k$$

is always positive and the phase fluctuations experience ultimate growth (linear in steady state)

Essentially RPA fields

- The amplitude variables are almost independent is a sense that for each M<<N modes the M-mode amplitude PDF is equal to the product of the one-mode PDF's up to $O(M/N)$ and $o(\epsilon^2)$ corrections.

$$\mathcal{P}_{j_1,j_2,\dots,j_M} = P^{(a)}_{j_1} P^{(a)}_{j_2} \dots P^{(a)}_{j_M} \left[1 + O(M/N) + O(\epsilon^2)\right]$$