A close one-term approximation to the highest Stokes wave on deep water

by

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Abstract

A remarkably accurate approximation to the profile of a limiting progressive gravity wave in water of infinite depth is given by the expression

$$y/L = A\cosh(x/L) \tag{0.1}$$

where L is the wavelength, x and y are horizontal and vertical coordinates and

$$A = 1/\left(\sqrt{3}\sinh\frac{1}{2}\right) = 1.1080. \tag{0.2}$$

This determines the wave steepness (H/L) as 0.14140 a proportional error of less than 0.3 percent (about ten times closer than previous approximations) and the phase speed $c/(gL)^{1/2}$ as 0.43511, which is accurate to within 0.2 percent. The entire surface profile is accurate to less than 0.7 percent. The corresponding particle velocities are found by a straightforward numerical integration. It is shown that this type of approximation cannot be made exact by the introduction of further parameters.

1. Introduction

A simple, one-term expression for the profile of the highest progressive gravity wave in deep water – the limiting Stokes wave – was proposed by one of the present authors (Longuet-Higgins 1973) namely

$$y = k^{-1} \ln \sec kX, \qquad -\pi/6 \leq kX \leq \pi/6, \tag{1.1}$$

when k is the wavenumber ($= 2\pi$/wavelength). The ratio of the crest-to-trough wave height to the wavelength L is then

$$(3/\pi) \ln \sec(\pi/6) = 0.1374 \tag{1.2}$$

an error of about 3 percent. This approximation has the advantage that it is associated with a simple, one-term expression for the velocity-potential ϕ.

The practical utility of such an approximation is of course lessened by the fact that steep gravity waves are unstable. (The form of the linear instability has been described by Longuet-Higgins and Tanaka 1997; for the subsequent nonlinear stages of growth see Longuet-Higgins and Dommermuth 1997.) Moreover, a breaking wave usually by-passes the 120° limiting form altogether and spills or plunges directly into the forward face of the wave. Nevertheless, the Stokes limiting wave is in many cases attained approximately for a limited period of time, so that for some purposes a simple, rough approximation to the flow can be useful. Like the linearised approximation, it is always well-defined.

In the following we shall describe an alternative but equally simple expression to equation (1.1), namely

$$y = AL\ \cosh(X/L), \qquad -\frac{1}{2} \leq X/L \leq \frac{1}{2} \tag{1.3}$$

where A is a certain constant. It will be shown that this describes the profile of the free surface at least ten times more accurately than equation (1.1). Moreover, although the corresponding velocity potential is not as simple as before, nevertheless the fluid velocities at every point on the free surface are expressed very simply. At points in the interior, the particle velocity may be found by performing a single integration which is easy to carry out numerically.

2. Basis for the approximation (1.3)

In order to save writing we shall henceforth take the wavelength L as unity. Thus

$$L = 1, \qquad k = 2\pi. \tag{2.1}$$

As shown by Stokes (1880) any progressive, irrotational gravity wave of limiting form has a sharp angle of 120° at its crests. Now a simple curve with a sharp angle of 120° at $x = 0$ is given by

$$y = C \ \times e^{-|x|}, \qquad C = 1/\sqrt{3} \tag{2.2}$$

see Figure 1. Let us superpose an infinite sequence of such sharp angles at the points $x = n - 1/2$ where n is any integer as in Figure 2. Thus let

$$y = C \sum_{n=-\infty}^{\infty} e^{-|x-n+1/2|}. \tag{2.3}$$

In the interval $-1/2 < x < 1/2$ the above series can be written

$$[e^{-(x+1/2)} + e^{-(x+3/2)} + e^{-(x+5/2)} + \ldots] + [e^{x-1/2} + e^{x-3/2} + e^{x-5/2} + \ldots] \tag{2.4}$$

that is

$$e^{-(x+1/2)}(1+ \ e^{-1} + e^{-2} + \ldots) + e^{(x-1/2)}(1 + e^{-1} + e^{-2} + \ldots)$$

$$= e^{-1/2}(e^x + e^{-x})/(1 - e^{-1}) \tag{2.5}$$

$$= (e^x + e^{-x})/(e^{1/2} - e^{-1/2})$$

$$= \frac{\cosh x}{\sinh \frac{1}{2}}. \tag{2.6}$$

So equation (2.3) may be written

$$y = A \cosh x. \tag{2.7}$$

where

$$A = \left(\sqrt{3} \sinh \frac{1}{2}\right)^{-1} = 1.107,955. \tag{2.8}$$

The corresponding ratio of crest-to-trough height to wavelength (unity) is then

$$A \left(\cosh \frac{1}{2} - 1 \right) = 0.141,404. \tag{2.9}$$

This is to be compared with the value 0.1410 obtained by accurate calculation (Williams 1981, Table 12a). The proportional error is 0.003 or 0.3 percent, which is about one-tenth of that involved in the approximation (1.1).

It will be convenient to shift the origin of y to the level of the wave trough by subtracting a constant, so the free surface is given by

$$Y = A\,(\cosh X - 1) \tag{2.10}$$

where $X = x$ and $Y = y - A$.

3. The complete profile

To see how closely the expression (2.10) fits the complete profile of the highest wave as computed accurately by Williams (1981), we may use the surface coordinates (x, y) given in his Table 12a on p. 180. (His calculations are for water of depth approximately 9 times the wavelength, so the profile is unaffected by the presence of the bottom.) His wavelength L equals 1.18482, with $g = 1$. The coordinates of the wave trough are

$$x_T = \frac{1}{2}L = 0.59241, \qquad y_T = 0.2568. \tag{3.1}$$

In our notation $X = |x - x_T|/L$; and $Y_1 = |y - y_T|/L$ is to be compared with $Y_2 = A(\cosh X - 1)$, as in Table 1 below. The difference $(Y_2 - Y_1)$ is given in the last column of Table 1. As will be seen, the difference is at most 0.0011, or 0.7 percent of the total wave height.

A comparison can also be made with the results of Schwartz (1974), presented graphically in his Figure 12 on p. 576. On enlarging his Figure by a factor 2 and carefully measuring it one may read his profile to almost the fourth decimal place. The differences $(Y_3 - Y_4)$ between

$$Y_3 = A|\cosh X - \cosh(1/2)| \tag{3.2}$$

and the corresponding values derived from Figure 12 are shown in Table 2 below. Clearly they are similar in magnitude to the differences $(Y_2 - Y_1)$ derived from Williams's calculations. Williams and Schwartz used completely different methods of calculation.

Table 1. Comparison of equation (2.10) with the surface coordinates (x, y) calculated by Williams (1981).

x	X	y	Y_1	Y_2	$(Y_2 - Y_1)$
.5924	.0000 (0)	.2568	.0000	.0000	.0000
.5438	.0410 (3)	.2558	.0008	.0009	.0001
.4949	.0823 (0)	.2525	.0036	.0038	.0002
.4456	.1239 (1)	.2471	.0082	.0085	.0003
.3955	.1661 (9)	.2393	.0148	.0153	.0005
.3443	.2094 (1)	.2288	.0236	.0244	.0008
.2914	.2540 (6)	.2153	.0350	.0360	.0010
.2360	.3008 (1)	.1983	.0494	.0505	.0011
.1763	.3512 (0)	.1764	.0679	.0690	.0011
.1085	.4084 (2)	.1469	.0928	.0937	.0009
.0000	.5000 (0)	.0897	.1410	.1414	.0004

Table 2. Comparison of the surface elevation in the highest wave as calculated by Schwartz (1974) and as given by equation (2.10).

x/L	X	Y_3	Y_4		$(Y_3 - Y_4)$
			cm		
0.00	0.50	0.0000	0.00	$\sim$ .0000	.0000
0.10	0.40	0.0516	4.25	.0522	.0006
0.20	0.30	0.0912	7.53	.0924	.0012
0.30	0.20	0.1192	9.79	.1201	.0009
0.40	0.10	0.1359	11.12	.1364	.0005
0.50	0.00	0.1414	11.49	.1412	.0002

4. Could equation (2.10) be exact?

The question naturally arises as to whether the expression (2.10), or perhaps some more general expression of the form

$$Y = B(\cosh \beta X - 1) \tag{4.1}$$

where β is a constant not necessarily unity and B is chosen so as to make the surface gradient at the crest equal to 30°, could possibly give an exact representation of the profile of the highest progressive wave ? This question is answered by a result due to Grant (1973) who showed that the surface elevation in the neighbourhood of the wave crest differs from the Stokes 120° angle corner-flow by a term of the form $C(i\chi)^{\nu}$, where C is a constant, $\chi = \phi + i\psi$ is the complex velocity-potential and ν is a root of the equation

$$\tan \frac{1}{2}\nu\pi = -(4 + 3\nu)/3\sqrt{3}\, \nu. \tag{4.2}$$

Longuet-Higgins and Fox (1977) pointed out that by substituting $\nu = -(\lambda + 1/3)$ equation (4.2) reduces to the simpler form

$$\left(\frac{1}{2}\lambda\pi\right) \tan \left(\frac{1}{2}\lambda\pi\right) = -\pi/(2\sqrt{3}). \tag{4.3}$$

We see that λ is transcendental, and hence also is ν.

On the other hand, any expression of the form (4.1) is regular analytically, near the crest. Hence it cannot represent the surface profile exactly. For this reason it does not seem worthwhile to pursue further any approximation such as (4.1) which involves the additional parameter β and so is less desirable than the simple expression (2.10).

5. The phase speed

The phase-speed c corresponding to the profile (1.3) may be found as follows. Suppose the wave travels to the left. Consider the motion in a frame of reference moving horizontally with the phase velocity $(-c, 0)$. Since the motion is steady we have by Bernoulli's equation

$$p + \frac{1}{2}\,q^2 + Y = N, \quad \text{constant} \tag{5.1}$$

where p and q denote the pressure and the fluid velocity and $Y = y - AL$. At the sharp wave crest we have $p = p_s$, where p_s denotes the pressure at the free surface, assumed constant, and $q = 0$. Hence

$$p_s + Y_c = N \tag{5.2}$$

where Y_c is the crest elevation. On the other hand at depths large compared to a wavelength the particle velocity relative to stationary axes vanishes, so relative to axes moving with horizontal velocity $-c$ we have $q = c$. Hence the pressure p_∞ is given by

$$p_\infty + \frac{1}{2}\,c^2 + Y = N. \tag{5.3}$$

From (5.2) and (5.3) it follows that

$$p_\infty + \frac{1}{2}\,c^2 + Y = p_s + Y_c. \tag{5.4}$$

But from the vertical flux of vertical momentum over one wavelength we find

$$p_\infty = p_s + (\overline{Y} - Y) \tag{5.5}$$

where $\overline{Y}$ is the mean value of the surface elevation. From equations (5.4) and (5.5) it follows that

$$\frac{1}{2}\,c^2 = Y_c - \overline{Y}. \tag{5.6}$$

Now from equation (1.3) we have

$$Y_c = A\Big(\cosh\frac{1}{2} - 1\Big) \tag{5.7}$$

and

$$\overline{Y} = A\int_{-1/2}^{1/2}(\cosh X - 1)dX = A\Big(2\sinh\left(\frac{1}{2}\right) - 1\Big). \tag{5.8}$$

So from equation (5.6)

$$\begin{aligned} \tfrac{1}{2}c^2 &= A\left(\cosh\frac{1}{2} - 2\sinh\frac{1}{2}\right) \\ &= 0.094658 \end{aligned} \tag{5.9}$$

and

$$c = 0.435106\ (gL)^{1/2} \tag{5.10}$$

in dimensional notation. For example when $g = 1$ and $L = 2\pi$, then

$$c = 1.090646 \tag{5.11}$$

compared to the value 1.092282 found by Williams (1985, p. 599). Thus equation (5.11) is accurate to within approximately 0.2 percent.

6. Particle velocities at the surface

Consider first the fluid velocities in a reference frame moving with horizontal velocity $-c$, as in Section 5.

At the free surface, since the pressure p is constant, we have from equations (5.1) and (5.2)

$$\frac{1}{2}\,q^2 = Y_c - Y \tag{6.1}$$

at any point on the surface. Hence

$$q = \sqrt{2}\,(Y_c - Y)^{1/2}. \tag{6.2}$$

We shall not need the velocity potential, only the fluid velocity. Write

$$Z = X + \mathrm{i}Y \tag{6.3}$$

so the components of fluid velocity are given by

$$\left.\begin{aligned} u &= \frac{q\,dX}{(dX^2 + dY^2)^{1/2}} \\ v &= \frac{q\,dY}{(dX^2 + dY^2)^{1/2}}. \end{aligned}\right\} \tag{6.4}$$

Together with the equation of the profile,

$$Y = A\,(\cosh X - 1) \tag{6.5}$$

equations (6.4) will determine u and v everywhere on the free surface.

7. Particle velocities in the interior

For the interior, we note that if $w = \phi + \mathrm{i}\psi$ is the complex velocity potential, then

$$\frac{dw}{dZ} = u - \mathrm{i}v \tag{7.1}$$

is an analytic function of Z within the fluid domain. Hence the velocities in the interior may be obtained from those at the surface by means of a contour integral, as follows.

Let us make the tranformation

$$\zeta = \exp(-2\pi \mathrm{i} Z) \tag{7.2}$$

where $Z = X + \mathrm{i}Y$ and $-1/2 \leq X \leq 1/2$, that is

$$\zeta = e^{2\pi Y}[\cos(2\pi X) - \mathrm{i}\sin(2\pi X)], \tag{7.3}$$

where $-\pi \leq \arg\zeta \leq \pi$; see Figure 4. The point $Z = 0$ transforms into $\zeta = 1$, and the point $Z = -\mathrm{i}\infty$ transforms into $\zeta = 0$. The free surface S transforms into the contour Γ having a sharp 120° corner at $\zeta = -e^{2\pi Y_c}$.

Now dw/dZ being analytic everywhere in the finite part of the Z-plane,

$$\frac{dw}{d\zeta} = \frac{dw}{dZ}\bigg/\frac{d\zeta}{dZ} \tag{7.4}$$

is an analytic function of ζ with a pole at $\zeta = 0$, and so by Cauchy's theorem (see Whittaker and Watson, 1940, c. 5)

$$H_0 \equiv \int_\Gamma \frac{dw/d\zeta}{\zeta - \zeta_0} d\zeta = 2\pi \mathrm{i}(R_1 + R_2) \tag{7.5}$$

where R_1 and R_2 denote the residues of the integrand at $\zeta = \zeta_0$ and at $\zeta = 0$ respectively. But

$$R_1 = (dw/d\zeta)_0 \tag{7.6}$$

while

$$R_2 = \lim_{\zeta\to 0}\left(\zeta\frac{dw/d\zeta}{\zeta-\zeta_0}\right) = \lim_{\zeta\to 0}\left(\zeta\frac{dw/dZ}{-\zeta_0\,d\zeta/dZ}\right) = \frac{c}{2\pi\mathrm{i}\,\zeta_0} \tag{7.7}$$

since the velocity at infinite depth is $(c, 0)$. So from equation (7.5)

$$\left(\frac{dw}{d\zeta}\right)_0 = \frac{1}{2\pi\mathrm{i}}(H_0 - c/\zeta_0). \tag{7.8}$$

Hence we have

$$u_0 - \mathrm{i}v_0 = \left(\frac{dw}{dZ}\right)_0 = -2\pi\mathrm{i}\zeta_0\left(\frac{dw}{d\zeta}\right)_0 = c - \zeta_0 H_0. \tag{7.9}$$

Now from (7.5)

$$H_0 = \int_S \frac{dw/dZ}{\zeta-\zeta_0}dZ = \int_S \frac{q(dX-\mathrm{i}dY)(dX+\mathrm{i}dY)}{(dX^2+dY^2)^{1/2}(\zeta-\zeta_0)} \tag{7.10}$$

by equation (6.4). Since $q = \sqrt{2}(Y_c - Y)$ this can be written

$$H_0 = \int_S \frac{\sqrt{2}(Y_c-Y)^{1/2}}{\zeta-\zeta_0}dS \tag{7.11}$$

where

$$dS = (dX^2+dY^2)^{1/2} = [1+(dY/dX)^2]^{1/2}dX. \tag{7.12}$$

Note that, in describing the contour S, deep water is on the left, i.e. X decreases. So, on carrying out the substitution for ζ we obtain

$$u_0 - \mathrm{i}v_0 = c - \sqrt{2}\int_{-1/2}^{1/2}\frac{(Y_c-Y)[1+(dY/dX)^2]^{1/2}}{1-\exp[2\pi\mathrm{i}(Z_0-Z)]}dX. \tag{7.13}$$

If Y represents the exact surface profile the above expression represents the exact, irrotational flow in the interior. If now we replace Y in (7.13) by the approximate expression (2.10) we obtain an approximation to the velocity field

$$u_0 - \mathrm{i}v_0 = c - (2A)^{1/2}\int_{-1/2}^{1/2}\frac{(\cosh 1/2 - \cosh X)^{1/2}(1+A^2\sinh^2 X)^{1/2}}{1-\exp[2\pi\mathrm{i}(Z_0-Z)]}dX \tag{7.14}$$

in which the error is of order $10^{-3}c$. Equation (7.14) also represents a flow in which the pressure at the free surface is constant and the flow is irrotational, with the possible exception of singularities at the free surface. There is in fact a very weak singularity at each wave crest $Y = Y_c,\ X = n \pm 1/2$.

8. Evaluation of the integral in (7.14)

Powerful and accurate integration subroutines are available for the evaluation of integrals such as (7.14). However it may be worth pointing out that the square-root singularities in the integrand at the two ends of the range of integration $X = \pm\frac{1}{2}$ can be removed by writing

$$\left(\frac{1}{2} - X\right)\left(\frac{1}{2} + X\right) = \lambda^2 \tag{8.1}$$

that is

$$\lambda^2 + X^2 = \frac{1}{4}, \qquad -\frac{1}{2} < X < \frac{1}{2} \tag{8.2}$$

which represents a semi-circle of radius $\frac{1}{2}$. So by Figure 5 we make the substitution

$$\left.\begin{aligned} X &= \ -\frac{1}{2}\cos\alpha \\ dX &= \ \frac{1}{2}\sin\alpha\ d\alpha \end{aligned}\right\} 0 \le \alpha \le \pi \tag{8.3}$$

and so from equation (7.14) we have finally

$$u_0 - \mathrm{i}v_0 = c - \left(\frac{1}{2}A\right)^{1/2} \int_0^{\pi} \frac{[\cosh(\frac{1}{2}) - \cosh X]^{1/2}[1 + A^2\sinh^2 X]^{1/2}}{1 - \exp[2\pi\mathrm{i}(Z_0 - Z)]} \sin\alpha\ d\alpha \tag{8.4}$$

where $Z_0 = X_0 + \mathrm{i}Y_0$, $Z = X + \mathrm{i}Y$, X is given by equation (8.3), and

$$Y = A\,[\cosh X - 1]. \tag{8.5}$$

The integration can now be calculated accurately by Simpson's rule. Figure 6 shows the particle velocity vectors in a frame of reference moving with the phase velocity $(-c, 0)$, and Figure 7 shows the velocity vectors in a stationary reference frame.

Note that the subsurface streamlines may be obtained easily by numerical integration of the velocity vectors.

Our conclusions are summarised in the Abstract.

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Diagram legends

Figure 1. Graph of $y = (1/\sqrt{3})e^{-|x|}$.

Figure 2. Graph of $y = (1/\sqrt{3})e^{-|x-n+1/2|}$, $n = -5, 4$.

Figure 3. Graph of $Y = A\,(\cosh X - 1)$.

Figure 4. Map of the ζ-plane, showing the contour Γ corresponding to the free surface S in Figure 3.

Figure 5. Definition of α.

Figure 6. Particle velocity vectors in a reference frame moving horizontally with the phase-speed c.

Figure 7. Particle velocity vectors in a stationary reference frame.

Figure 8. Vertical profile of the horizontal particle velocities in Figure 7, (a) beneath a wave crest and (b) beneath a wave trough.

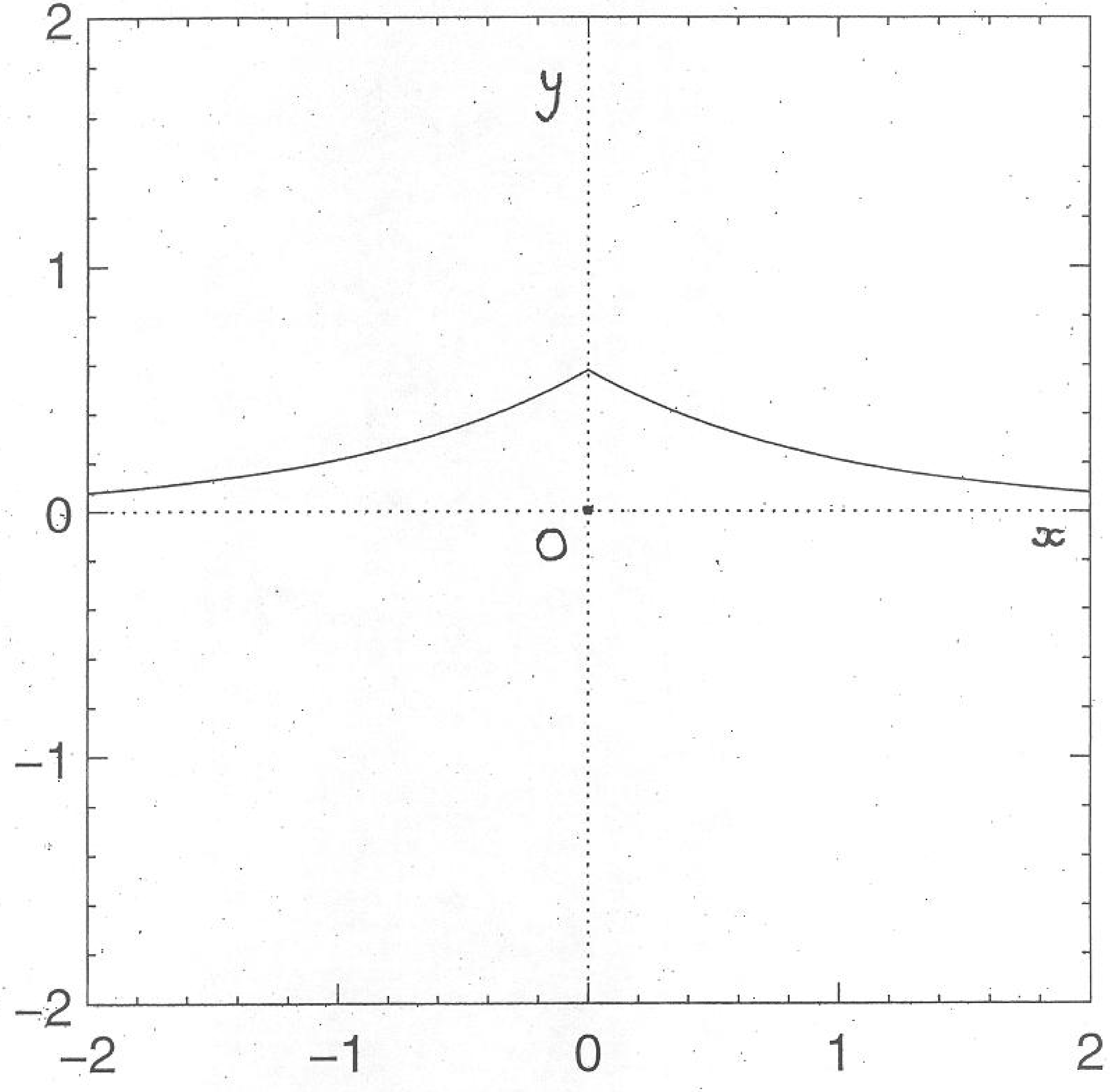


Figure 1.

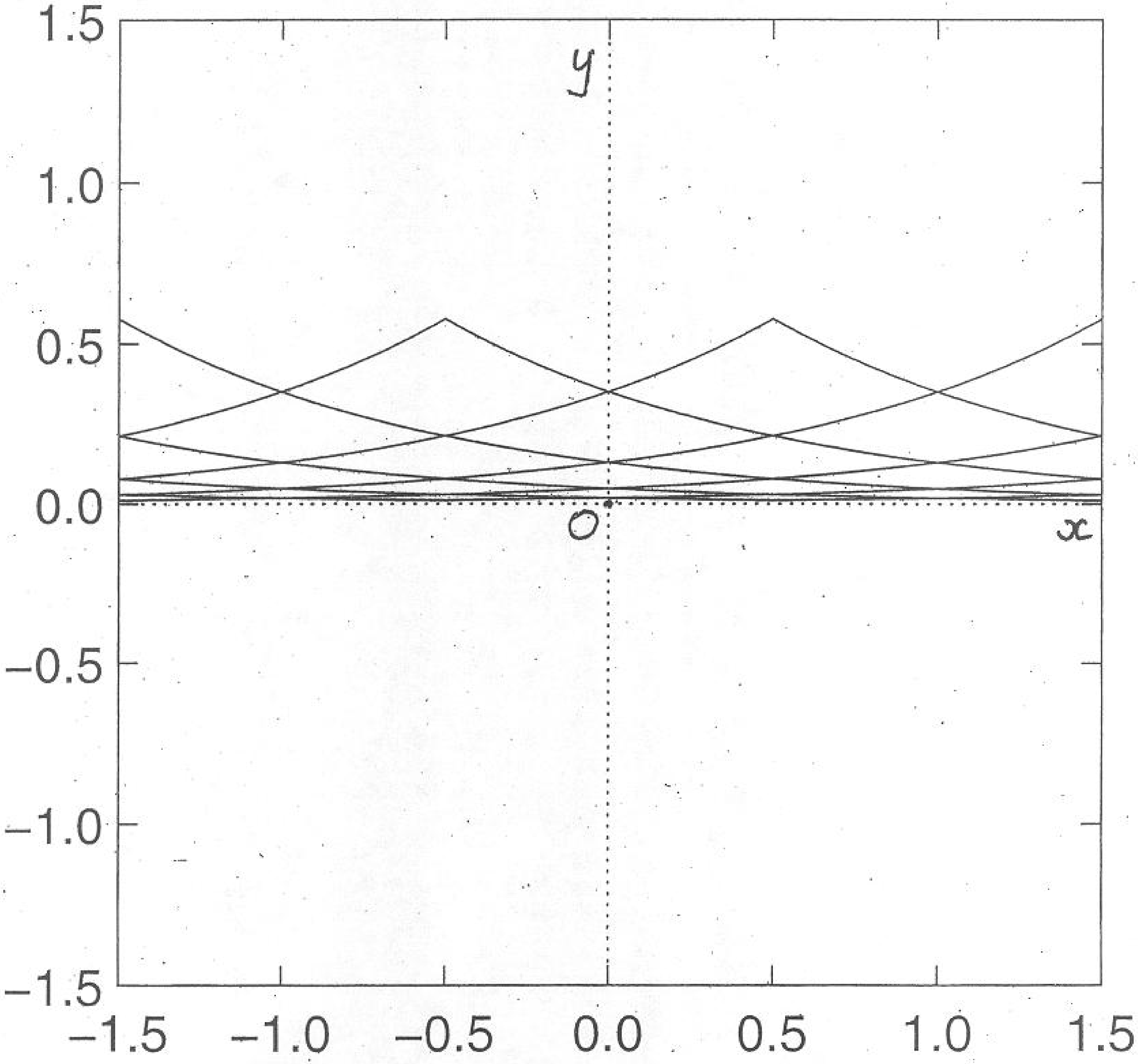
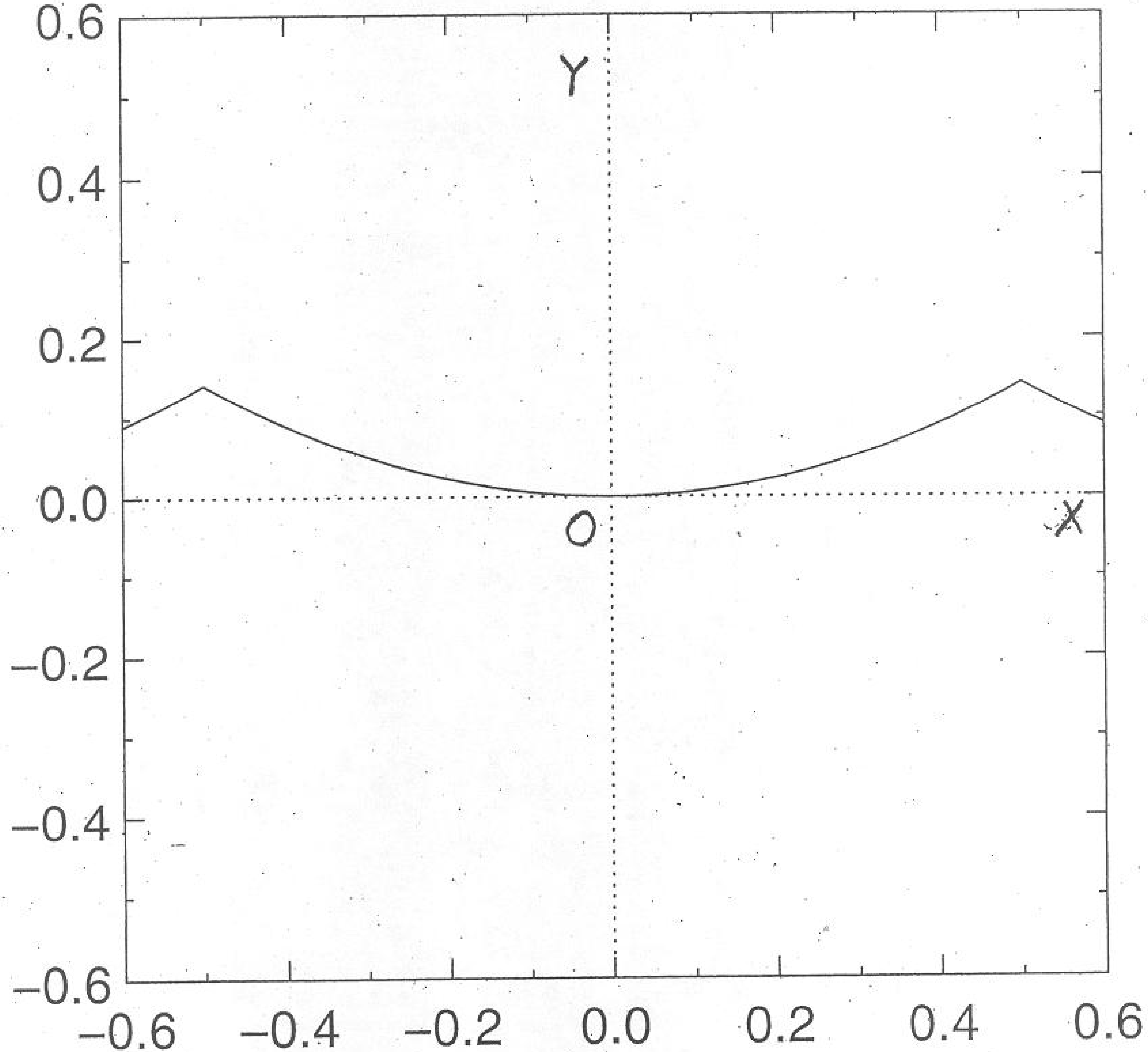


Figure 2.



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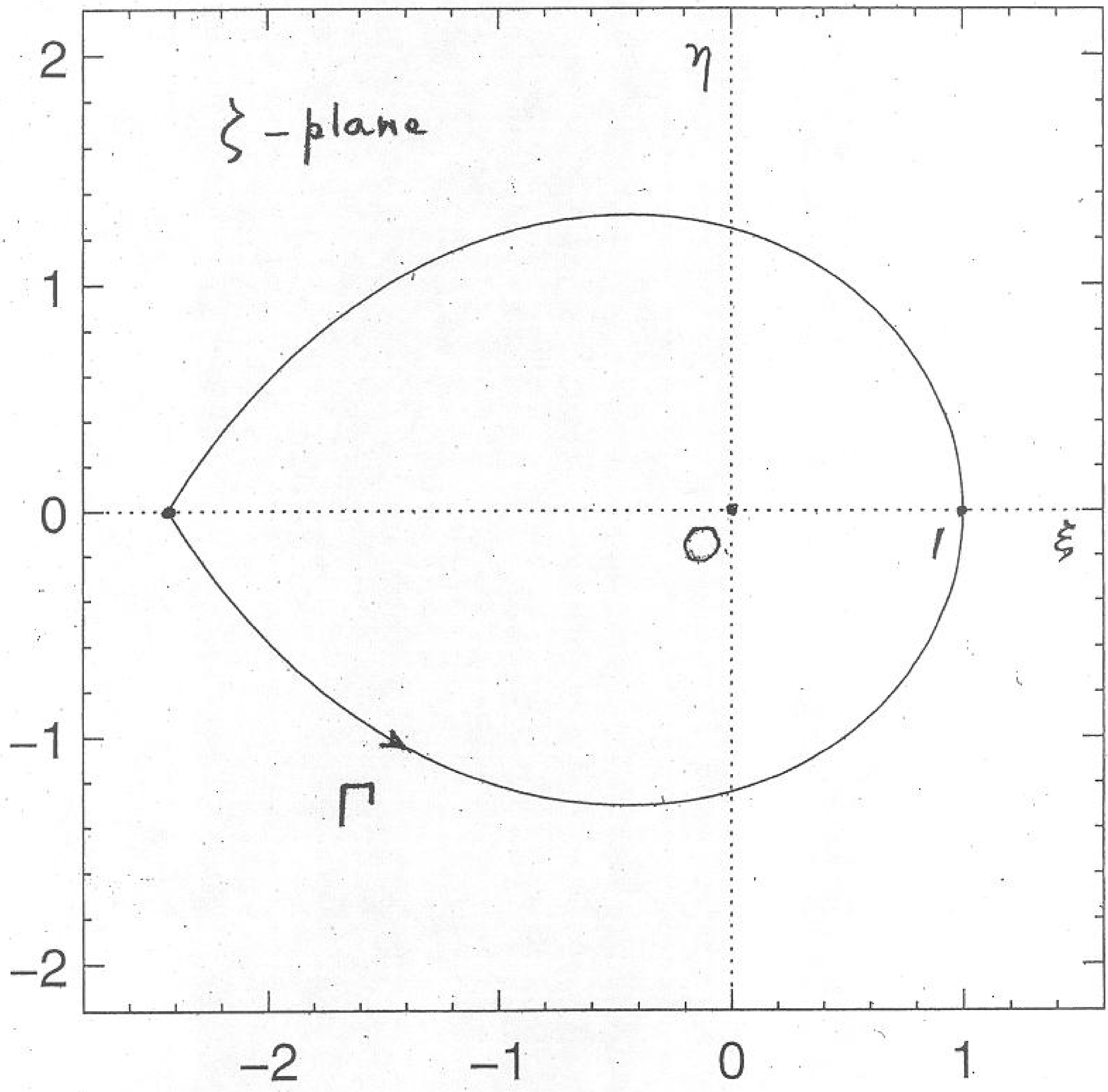

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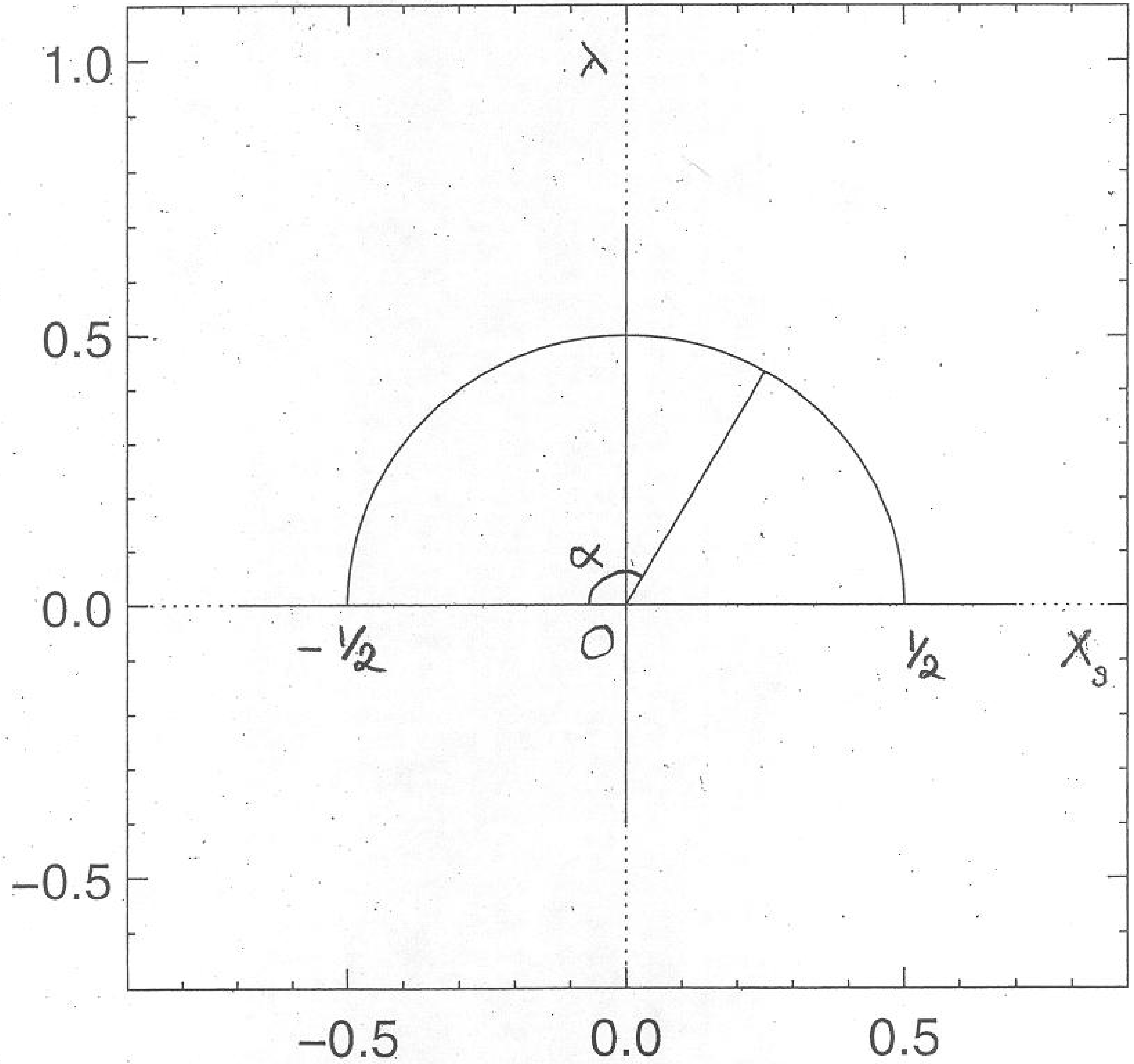
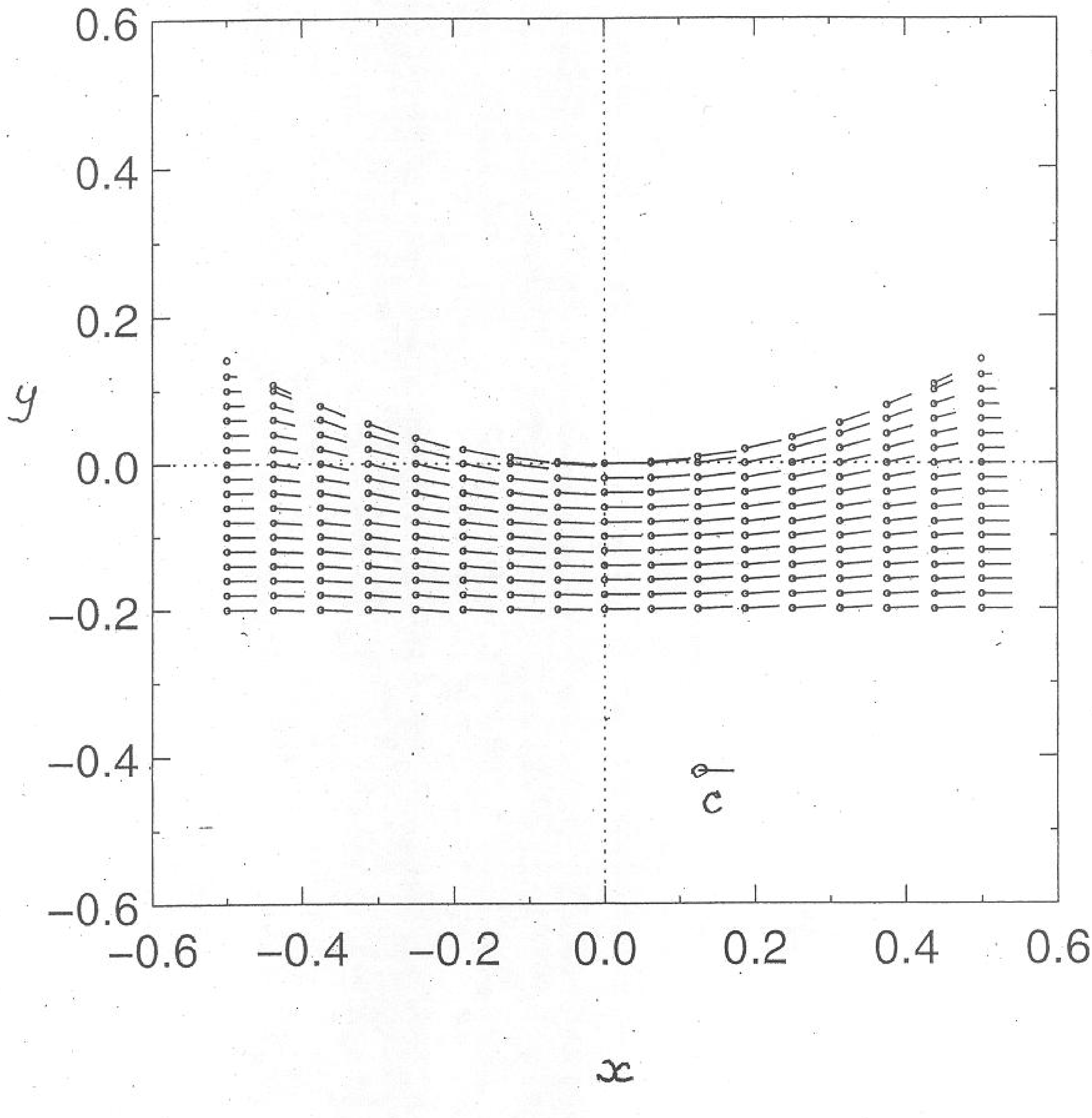


Figure 5.



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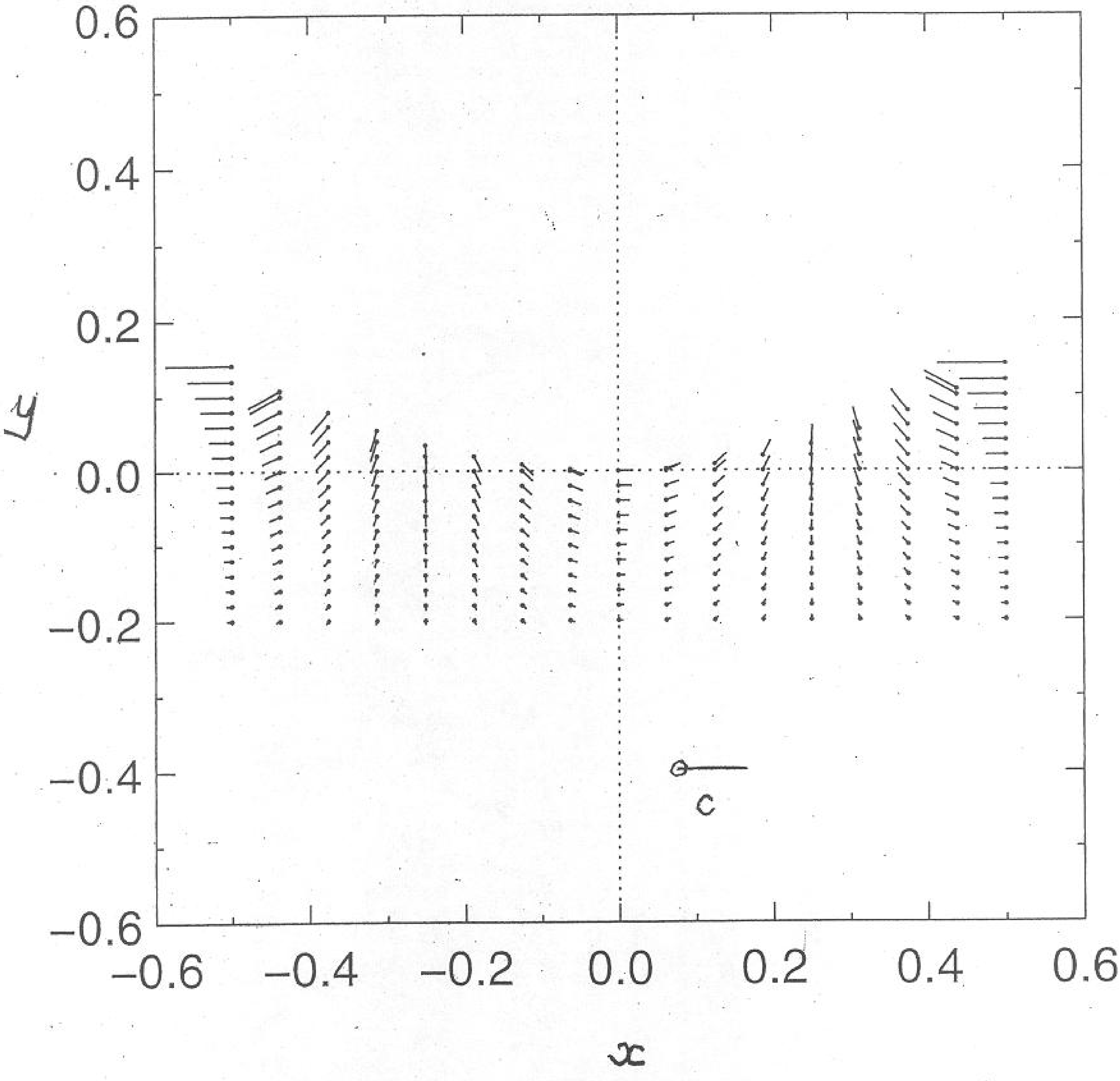

Figure 7.

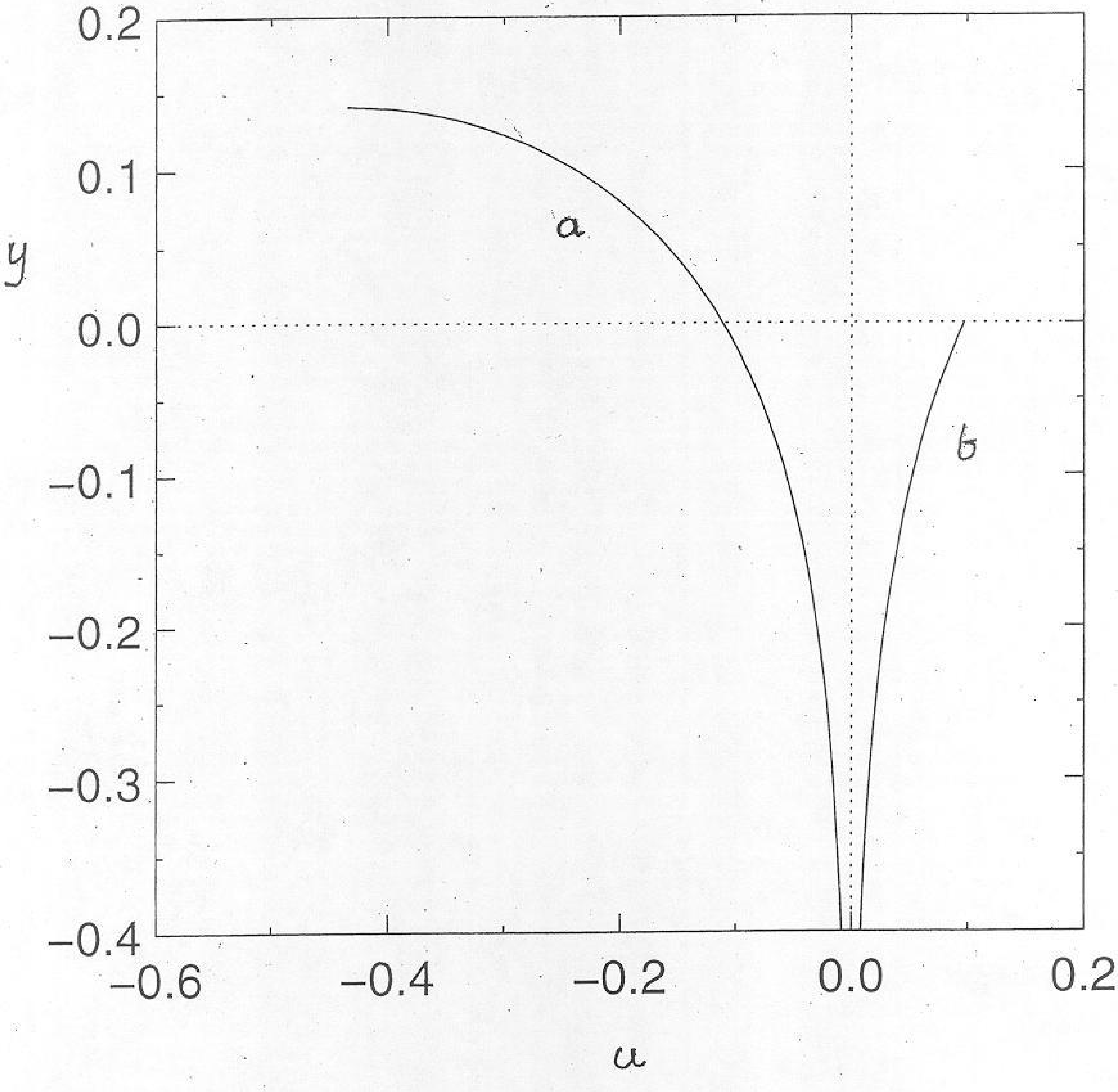

Figure 8.