

# Nonlinear focussing and shallow water

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## INTRODUCTION

The programme of the talk is as follows.

- **FREAK WAVES GENERATION**

On the open ocean extreme waves are generated by **nonlinear focussing**, a process that also causes the Benjamin-Feir Instability. Deviations from the normal probability distribution depend on both **resonant and non-resonant** interactions: Phillips' figure of eight not relevant for this aspect.

- **SHALLOW WATER**

In shallow water finite amplitude waves generate a current and a mean surface elevation. Stabilizes the Benjamin-Feir Instability and **nonlinear focussing** disappears when  $kh < 1.363$ . This is a well-known phenomena first found by Benjamin (1967) and Whitham (1974). It is shown that the **Zakharov equation** accounts for the stabilizing effects of wave-induced current and mean sea surface elevation.



- **CONSEQUENCES FOR WAVE PREDICTION**

Because of the change of stability there is a remarkable reduction of the nonlinear transfer for  $kh$  around 1.363. This has not been noted before, it is not given by the Hasselmann form of  $S_{nl}$ , but it needs to be included! Freak wave occurrence is considerably reduced by the effect of the wave-induced current.

## FREAK WAVES GENERATION

In linear theory there is no interaction between ocean waves. Focussing of wave energy only occurs when the phases of the waves are favourable (**constructive interference**). Gives at best a doubling of wave height. Accordingly, the probability distribution function for the surface elevation  $\eta$  is given by the Gaussian distribution.

However, the situation for **nonlinear** waves is entirely different, because now there is the possibility of wave-wave interaction. Thus, a wave may borrow energy from its neighbours. Because of this extra focussing wave height may become at most 3 times as large in 1D, while in 2D it becomes 4.5-5 times as large as the average wave height.

As a consequence, for nonlinear waves the surface elevation distribution is rather different, in particular for the extremes.



## STOCHASTIC APPROACH

Starting point is the **Zakharov equation**, which describes the evolution of the complex amplitude  $a(\vec{k})$  of the free gravity waves:

$$\frac{\partial a_1}{\partial t} + i\omega_1 a_1 = -i \int d\vec{k}_{2,3,4} T_{1,2,3,4} a_2^* a_3 a_4 \delta_{1+2-3-4},$$

where  $\vec{k}$  is the wave number and  $\omega = \sqrt{gk}$ .  $T_{1,2,3,4}$  is a complicated function of frequency and wavenumber, and has a number of symmetries which guarantee that the system is **Hamiltonian**.

**Note:** The bound waves are eliminated from the Hamilton equations for the action variable  $A$  by a canonical transformation of the type  $A = A(a, a^*)$ .

In wave forecasting we are interested in predicting quantities such as the second moment

$$B_{1,2} = \langle a_1 a_2^* \rangle,$$

where angle brackets denote an ensemble average. Following methods employed in Statistical Mechanics (Liouville  $\rightarrow$  Boltzmann) one obtains from the deterministic Zakharov equation an equation for the action density  $N$ . Two assumptions are made:

- Because of nonlinearity second moment is coupled to fourth moment, etc. Closure achieved by the assumption that the sea state is close to a **Gaussian**, because nonlinearity is weak. For example, the fourth moment is

$$\langle a_j a_k a_l^* a_m^* \rangle = B_{j,l} B_{k,m} + B_{j,m} B_{k,l} + D_{j,k,l,m},$$

where  $D$  is the so-called fourth cumulant, which vanishes for a Gaussian sea state. For weakly nonlinear waves  $D$  is small, but finite, and this enables one to close the hierarchy of equations.



- The second assumption we make is that of a **homogeneous** wave field. For this the two point correlation function  $\langle \eta(\vec{x}_1)\eta(\vec{x}_2) \rangle$  depends only on the distance  $\vec{x}_1 - \vec{x}_2$ . As a consequence, the second moment becomes

$$B_{i,j} = N_i \delta(\vec{k}_i - \vec{k}_j),$$

where  $N_i$  is the spectral action density.

As a result one finds that nonlinearity gives rise to deviations from the Normal (Gaussian) distribution, while the action density obeys

$$\frac{\partial}{\partial t} N_4 = 4 \int d\vec{k}_{1,2,3} T_{1,2,3,4}^2 \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) R_i(\Delta\omega, t) \\ \times [N_1 N_2 (N_3 + N_4) - N_3 N_4 (N_1 + N_2)],$$

where  $\Delta\omega = \omega_1 + \omega_2 - \omega_3 - \omega_4$ . This evolution equation is usually called the Boltzmann equation.

Note there are now two timescales implied by

$$R_i(\Delta\omega, t) = \sin(\Delta\omega t) / \Delta\omega$$

- short times:  $\lim_{t \rightarrow 0} R_i(\Delta\omega, t) = t$ , hence  $T_{NL} = O(1/\epsilon^2 \omega_0)$ , the Benjamin-Feir timescale, corresponding to **non-resonant** interactions.
- large times:  $\lim_{t \rightarrow \infty} R_i(\Delta\omega, t) = \pi \delta(\Delta\omega)$ , corresponding to **resonant** wave-wave interactions, hence  $T_{NL} = O(1/\epsilon^4 \omega_0)$  (Hasselmann, 1962).

Deviations from Normality are most conveniently expressed by means of the kurtosis

$$C_4 = \langle \eta^4 \rangle / 3 \langle \eta^2 \rangle^2 - 1,$$



It becomes

$$C_4 = \frac{4}{g^2 m_0^2} \int d\vec{k}_{1,2,3,4} T_{1,2,3,4} \delta_{1+2-3-4} (\omega_1 \omega_2 \omega_3 \omega_4)^{\frac{1}{2}} \times R_r(\Delta\omega, t) N_1 N_2 N_3,$$

where

$$R_r(\Delta\omega, t) = \frac{1 - \cos(\Delta\omega t)}{\Delta\omega}.$$

Hence for short times

$$\lim_{t \rightarrow 0} R_r(\Delta\omega, t) = \frac{1}{2} \Delta\omega t^2$$

while for large times

$$\lim_{t \rightarrow \infty} R_r(\Delta\omega, t) = P/\Delta\omega$$

In other words, for short times resonant interactions do not contribute to the kurtosis (because  $R_r$  vanishes on the resonant surface) while for large times the kurtosis is determined by both **resonant** and **non-resonant** interactions. So **Phillips' figure of eight** not relevant in this context.



- **Remark 1**

The expression for the kurtosis is too involved in an operational context. However, for Gaussian-shaped spectra in the narrow band approximation the kurtosis shows a particularly simple form. For large times I find

$$C_4 = \frac{\pi}{3\sqrt{3}} \times BFI^2,$$

hence the kurtosis depends on the square of the BF index. Here, the **Benjamin-Feir Index** is defined as

$$BFI = \epsilon\sqrt{2}/\sigma'_\omega,$$

where  $\sigma'_\omega = \sigma_\omega/\omega_0$  is the relative width of the frequency spectrum and  $\epsilon = (k_0^2 \langle \eta^2 \rangle)^{\frac{1}{2}}$  is an integral measure of wave steepness (with  $\langle \eta^2 \rangle$  the average surface elevation variance and  $k_0$  the peak wave number).

- **Remark 2: Bound Waves**

In good approximation one may assume that long ocean waves have a narrow spectrum. In that case the surface elevation is given by the following unsteady Stokes expansion:

$$\eta = a\left(1 + \frac{1}{8}k_0^2 a^2\right) \cos \theta + \frac{1}{2}k_0 a^2 \cos 2\theta + \frac{3}{8}k_0^2 a^3 \cos 3\theta + \frac{1}{\omega} \frac{\partial a}{\partial t} \sin \theta.$$

Assuming Gaussian statistics for amplitude  $a$  and phase  $\theta$  one finds for the skewness

$$\mu_3 = \frac{\langle \eta^3 \rangle}{\langle \eta^2 \rangle^{3/2}} = 3\epsilon, \quad (1)$$

while the kurtosis becomes (Vinje, 1989)

$$C_4 = 8\epsilon^2 + \frac{\pi}{3\sqrt{3}} \times BFI^2.$$

Therefore, the contribution from wave-wave interactions dominates the one from the bound waves for narrow-band spectra:  $\sigma'_\omega < \sqrt{\pi/12\sqrt{3}} \simeq 0.39$ .



## SHALLOW WATER EFFECTS

In shallow water the wave-induced current and mean surface elevation have a stabilizing effect in such a way that for  $k_0 h_0 < 1.363$  the Benjamin-Feir Instability disappears and there is no focussing (Benjamin, 1967; Whitham, 1974). This is understood most easily from Whitham's variational approach. The **nonlinear** dispersion relation on a current  $\beta$  is

$$\frac{(\omega - k\beta)^2}{gk \tanh kh} = 1 + \frac{9T_0^4 - 10T_0^2 + 9}{4T_0^4} \frac{k^2 E}{g},$$

with  $T_0 = \tanh kh_0$  and  $E = ga^2/2$ . This dispersion relation is accompanied by equations for the current  $\beta$  and mean elevation  $b = h - h_0$ . Whitham finds

$$b = -\frac{h_0}{c_S^2 - v_g^2} \frac{S}{h_0}, \quad c_S^2 = gh_0,$$

with  $S$  the radiation stress,

$$S = \left( \frac{2v_g}{c_0} - \frac{1}{2} \right) E,$$



while

$$U = \beta + \frac{E}{c_0 h_0} = \frac{v_g}{h_0} b$$

Linearizing in  $b$ , the dispersion relation becomes

$$\omega = \omega_0 + \Omega_2(k) \frac{k^2 E}{c_0}, \omega_0^2 = gkT_0,$$

where

$$\Omega_2(k) = \frac{9T_0^4 - 10T_0^2 + 9}{8T_0^3} - \frac{1}{k_0 h_0} \left\{ \frac{(2v_g - c_0/2)^2}{c_S^2 - v_g^2} + 1 \right\}.$$

The curly bracketed term is positive definite and leads to **stabilization** of the Benjamin-Feir instability. At  $kh_0 = 1.363$ ,  $\Omega_2$  vanishes.



The important point to note is that the Zakharov equation represents the effects of wave-induced current and mean surface elevation. Taking a single wave,  $a = a_0 \delta(\vec{k} - \vec{k}_0)$ , the Zakharov equation becomes

$$\frac{\partial a_0}{\partial t} + i\omega_0 a_0 = -iT_{0,0,0,0}|a_0|^2 a_0$$

Solving this with the Ansatz  $a_0 = \rho e^{-i\Omega t}$  the result is

$$\Omega = \omega_0 + T_{0,0,0,0}|a_0|^2.$$

In order to be able to compare with results obtained by Whitham (1974) introduce the energy  $E$  of a wave train:  $E = \omega_0 |a_0|^2$  Writing the dispersion relation as

$$\Omega = \omega_0(k_0) + \Omega_2(k_0) \frac{k_0^2 E}{c_0}$$

one finds for  $\Omega_2$ ,

$$\Omega_2 = T_{0,0,0,0}/k_0^3,$$

thus we need  $T_{0,0,0,0}$ .

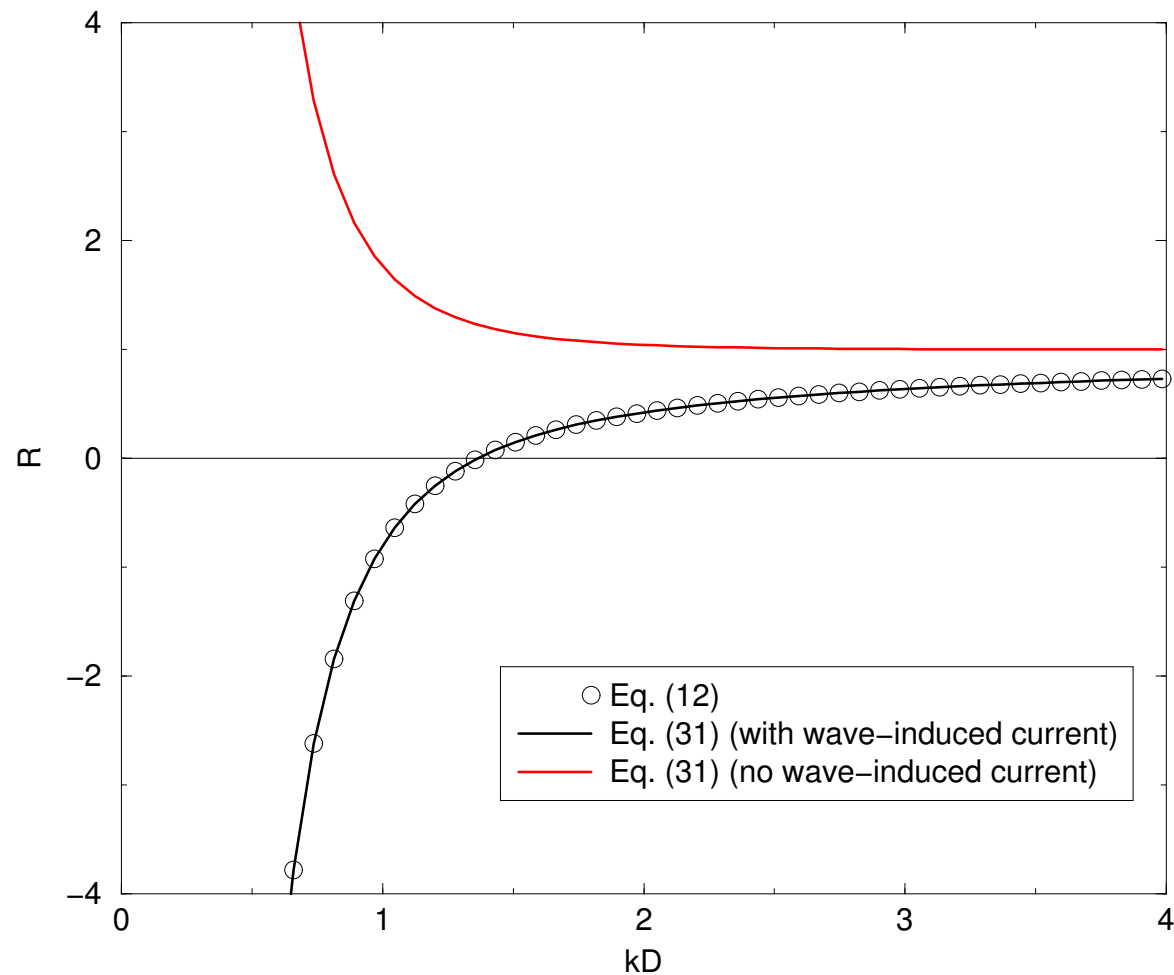


It is not a trivial task to evaluate  $T_{0,0,0,0}$  because of apparent singularities.  $T_{0,0,0,0}$  consists of three parts, a direct interaction a virtual interaction and a wave-induced current/surface elevation part.

$$\begin{aligned}
T_{1,2,3,4} = & W_{1,2,3,4} \\
& - V_{1,3,1-3}^{(-)} V_{4,2,4-2}^{(-)} \left[ \frac{1}{\omega_3 + \omega_{1-3} - \omega_1} + \frac{1}{\omega_2 + \omega_{4-2} - \omega_4} \right] \\
& - V_{2,3,2-3}^{(-)} V_{4,1,4-1}^{(-)} \left[ \frac{1}{\omega_3 + \omega_{2-3} - \omega_2} + \frac{1}{\omega_1 + \omega_{4-1} - \omega_4} \right] \\
& - V_{1,4,1-4}^{(-)} V_{3,2,3-2}^{(-)} \left[ \frac{1}{\omega_4 + \omega_{1-4} - \omega_1} + \frac{1}{\omega_2 + \omega_{3-2} - \omega_3} \right] \\
& - V_{2,4,2-4}^{(-)} V_{3,1,3-1}^{(-)} \left[ \frac{1}{\omega_4 + \omega_{2-4} - \omega_2} + \frac{1}{\omega_1 + \omega_{3-1} - \omega_3} \right] \\
& - V_{1+2,1,2}^{(-)} V_{3+4,3,4}^{(-)} \left[ \frac{1}{\omega_{1+2} - \omega_1 - \omega_2} + \frac{1}{\omega_{3+4} - \omega_3 - \omega_4} \right] \\
& - V_{-1-2,1,2}^{(+)} V_{-3-4,3,4}^{(+)} \left[ \frac{1}{\omega_{1+2} + \omega_1 + \omega_2} + \frac{1}{\omega_{3+4} + \omega_3 + \omega_4} \right].
\end{aligned}$$

A definite numerical and analytical answer is found by taking limits in such a way that one stays on the resonance surface  $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$  (R. Gorman, 2003). The analytical result is identical to the one found by Whitham (1974). Also, mean surface elevation and mean current are identical to the Whitham result (1974).

## Depth dependent correction factor Zakharov Equation



Depth dependence of numerical and analytical narrow-band approximation of the nonlinear transfer coefficient normalized with the deep water value. The effect of the wave-induced current and mean surface elevation is shown as well.

## CONSEQUENCES FOR WAVE PREDICTION

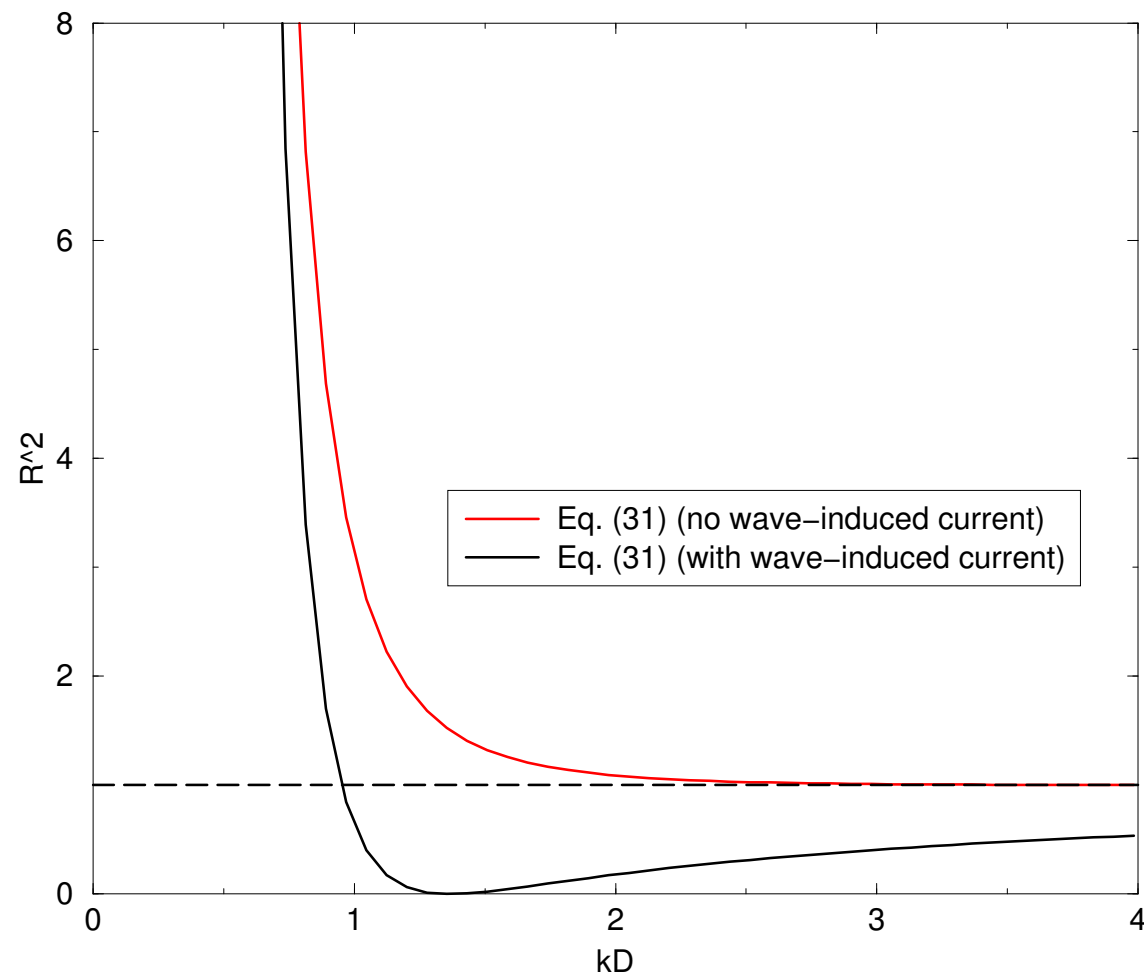
The threshold for instability at  $k_0h = 1.363$  has important consequences for wave modelling in shallow waters of **intermediate** depth. The reason is that for these dimensionless depths there is a considerable reduction of the nonlinear transfer and hence the shape of the wave spectrum is only determined by the balance of wind input and dissipation.

In order to test this conjecture I simulated the evolution of the wave spectrum by performing Monte Carlo Forecasting of Zakh. Eq. (cf. Janssen, 2003) for a number of cases namely  $k_0h = 1.363/2$ ,  $k_0h = 1.363$  and  $k_0h = 3 \times 1.363$ . The size of the ensemble is 500, while the Benjamin Feir Index equals 1.

### RESULTS

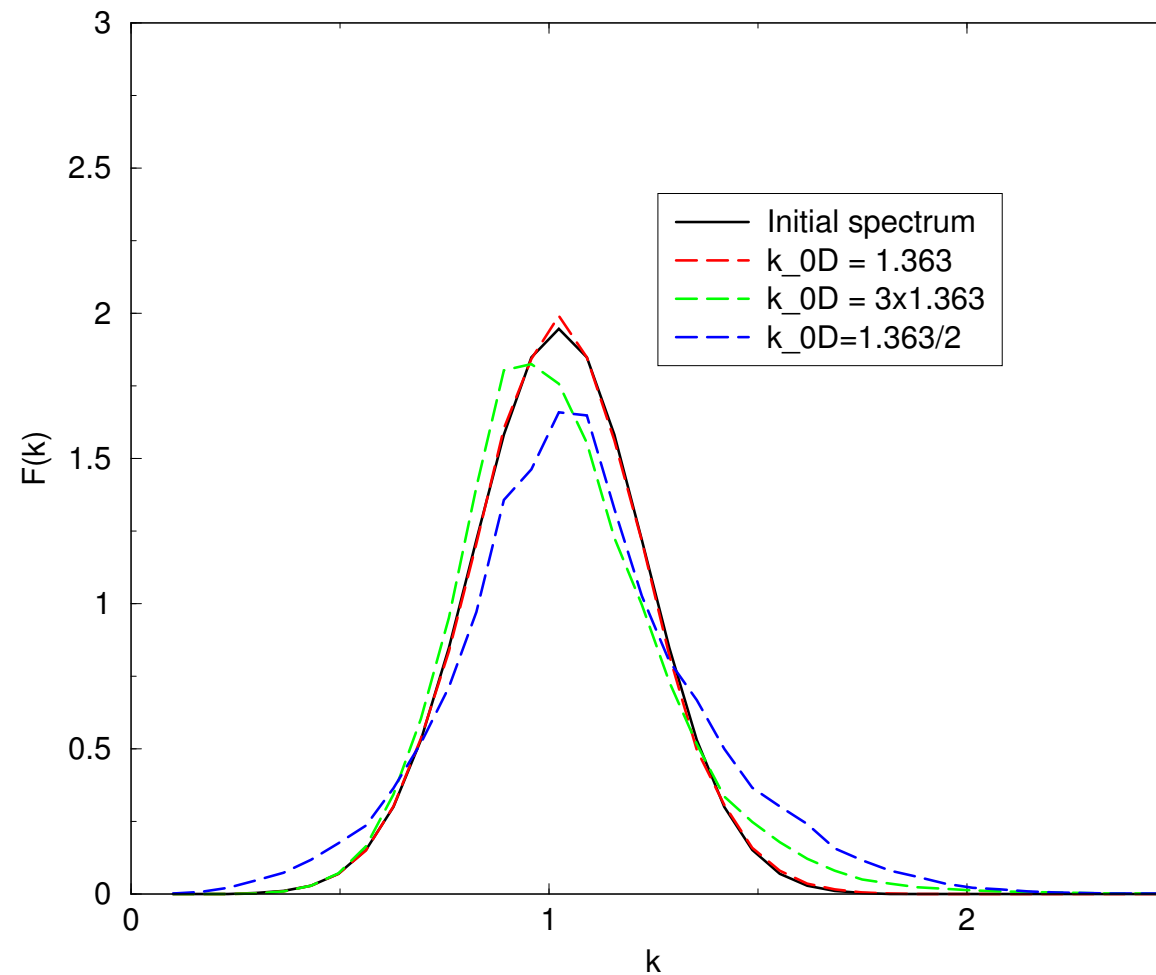
- show spectral evolution
- evolution of kurtosis

## Depth dependent correction factor Kinetic Equation



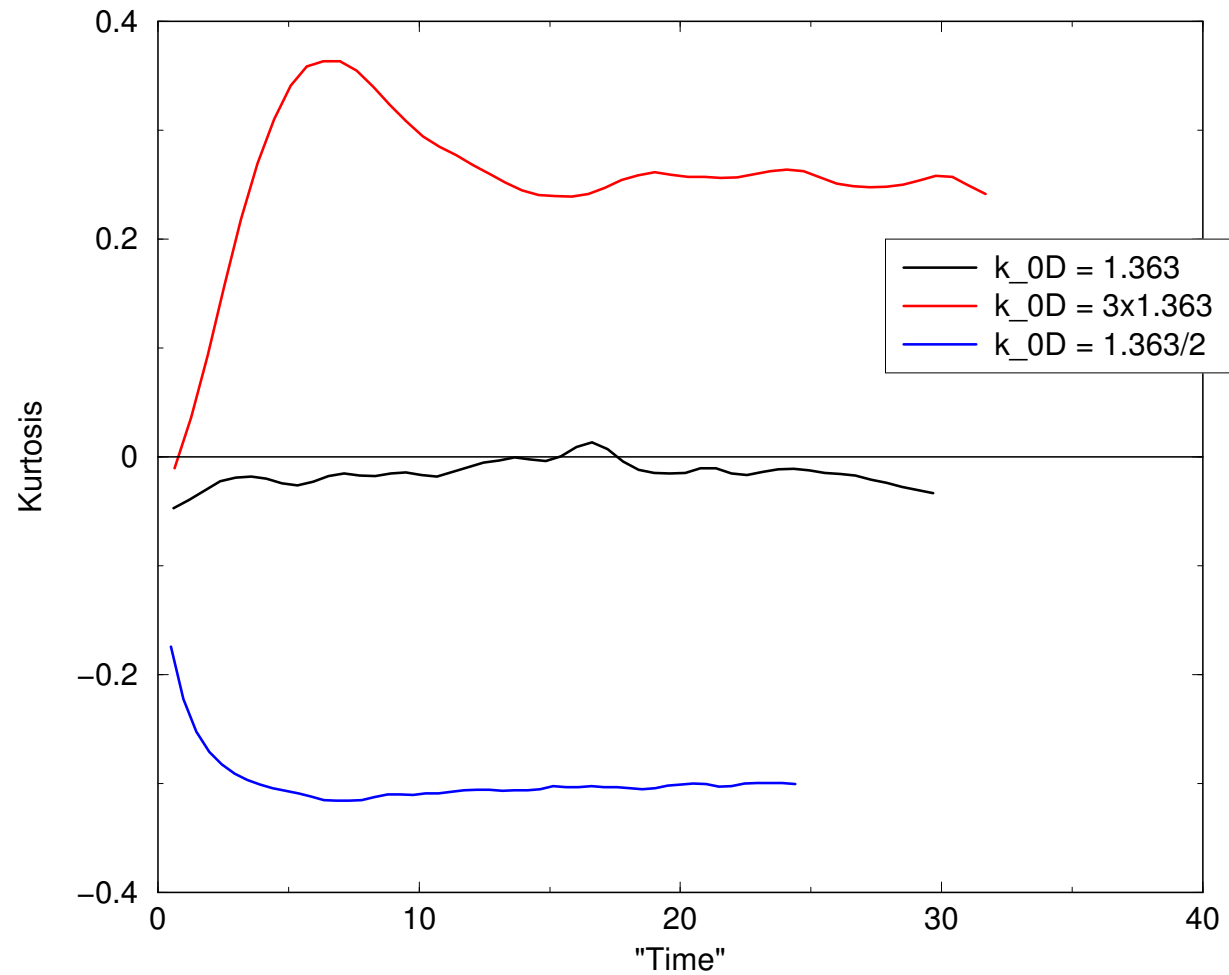
Depth dependence of the square of the nonlinear transfer coefficient in the narrow-band approximation in comparison with the case when wave-induced effects are removed.

## Wave number spectrum for BFI = 1



Spectral evolution of shallow-water case ( $k_0D = 1.363/2$ ), an intermediate depth case ( $k_0D = 1.363$ ) and a 'deep'-water case ( $k_0D = 3 \times 1.363$ ).

## Evolution of Kurtosis



Time evolution of kurtosis for  $BFI = 1$ .

## CONCLUSIONS

- Initially only **non-resonant** interactions contribute to the formation of freak waves, if deviations from the normal distribution are used as a measure for extreme sea states. At final time both resonant and nonresonant interactions contribute.
- In waters of **intermediate** depth nonlinear focussing is absent. Example: the Draupner case with  $k_0 h = 1.17$
- The **Hasselmann** transfer in intermediate depth waters is incorrect! In these circumstances I expect a balance between windinput and dissipation, but, of course, only for  $k_0 h$  around 1.363.