Bounds for Stop-Loss Premiums of Life Annuities with Random Interest Rates

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8. Application to life annuities with stochastic interest rates
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Introduction: problem description and motivation

- life annuity: a series of periodic payments where each payment will actually be made only if a designated life is alive at the time the payment is due

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- life annuity: a series of periodic payments where each payment will actually be made only if a designated life is alive at the time the payment is due
- notation:

  T_x: future lifetime of (x) (a person aged x years)

  $\hookrightarrow G_x(t) = \text{Pr}[T_x \leq t] = {}_tq_x, \; t \geq 0 \quad (G_x^{-1}(1) = \omega - x)$

  $K_x = \lfloor T_x \rfloor$: curtate future lifetime of (x)

  $\hookrightarrow \text{Pr}(K_x = k) = \text{Pr}(k \leq T_x < k+1) = {}_{k+1}q_x - {}_kq_x = {}_{k|}q_x, \; k = 0, 1, \ldots$

- a discrete whole life annuity-immediate:

$$a_{\overline{K_x}|} = \mathbb{S}_x^{policy} = \sum_{i=1}^{K_x} e^{Z_i} = \sum_{i=1}^{\lfloor \omega - x \rfloor} I_{(T_x > i)} e^{Z_i}$$

e^{Z_i} random discount factor over the period $[0, i]$

1 per annum payable at beginning of each year

- a continuous whole life annuity:

$$a_{\overline{T_x}|} = \int_0^{T_x} e^{-r(\tau)} d\tau$$

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Why stochastic interest rates?

- durations of contracts are typically very long (often 30 years or even more) uncertainty about future rates of return becomes very high

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Why stochastic interest rates?

- durations of contracts are typically very long (often 30 years or even more) uncertainty about future rates of return becomes very high
- financial and investment risk — unlike the mortality risk — cannot be diversified with an increase in the number of policies

Actuarial literature on stochastic interest rates

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  two-sided risk measure $\Rightarrow$ over- and underestimation of reserve in case of skewed distribution and tail properties of distribution not emphasized
- appropriate risk measures: Value-at-Risk, Tail Value-at-Risk or Expected Short Fall on based stop-loss premium
  stop-loss premium with retention d:
  $\pi(X, d) = \mathrm{E}[(X - d)_+] = \mathrm{E}[\max(0, X - d)]$

Model assumptions

discrete whole life annuity-immediate:

$$a_{\overline{K_x}|} = \mathbb{S}_x^{policy} = \sum_{i=1}^{K_x} e^{Z_i}$$

- mortality process: Makeham's model

  the number of persons alive at age x:

  $$l_x = as^x g^{c^x}$$

  with prespecified parameters $a > 0, 0 < s < 1, 0 < g < 1$ and $c > 1$ respectively for men and women

  number of newborns: $l_0 = 1\,000\,000$

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$$a_{\overline{K_x}|} = \mathbb{S}_x^{policy} = \sum_{i=1}^{K_x} e^{Z_i}$$

- discounting process: Brownian motion (motivation: Cesari & Cremonini (2003))

$$e^{Z_i} = e^{-Y(i)} := e^{-(Y_1+\cdots+Y_i)}$$

  – rv's Y_i represent stochastic continuous compounded rate of return over period $[i-1, i]$ and $e^{-Y(i)}$ is random discount factor over period $[0, i]$

  – yearly returns Y_i are i.i.d. normally distributed with mean μ and volatility σ
  $\Rightarrow Y(i) \sim i\mu + \sigma B(i)$
  with $B(i)$ standard Brownian motion independent of K_x,
  μ constant force of interest

$Y(i)$ strongly dependent!

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- Why is life simple using comonotonic approximations?
- Are these approximations accurate?

General treatment

- stop-loss premium with retention d:
  $\pi(X,d) = \mathrm{E}[(X-d)_+] = \mathrm{E}[\max(0, X-d)]$
- computation: not always straightforward!
  - if only partial information about the claim size distribution is available (e.g. De Vylder and Goovaerts (1982), Jansen et al. (1986), Hürlimann, (1996, 1998) among others)
  - in the case of a sum of random variables $\mathbb{S} = X_1 + \cdots + X_n$ with unknown dependency structure

$$\Downarrow$$

* convex upper and lower bounds for stop-loss premiums of sums of dependent random variables [Kaas et al. (2000), Dhaene et al. (2002)]
* ideas of splitting up the expectation in an exact part and the rest [e.g. Curran (1994)]
* ideas of conditioning as in Curran (1994) and Rogers and Shi (1995)

$\hookrightarrow$ leads to easy analytical computable bounds

Convex order and comonotonic risks

- Definition: Consider two random variables X and Y. Then X is said to precede Y in the convex order sense, notation $X \leq_{cx} Y$, if and only if

$$\mathrm{E}[X] = \mathrm{E}[Y] \text{ and } \mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+] \qquad \forall\, d$$

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- Comonotonicity is a very strong positive dependence structure
  $\longrightarrow$ each two possible outcomes $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ of $\vec{X} = (X_1, \cdots, X_n)$ are ordered componentwise

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Characterizations:

(a) for $U \sim \mathrm{uniform}(0,1)$ we have

$$\vec{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)),$$

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(b) $\exists$ a random variable Z and non-decreasing functions $f_1, f_2, \ldots, f_n$, (or non-increasing functions) such that

$$\vec{X} \stackrel{d}{=} (f_1(Z), f_2(Z), \ldots, f_n(Z)).$$

Comonotonic bounds for a stop-loss premium

Upper and lower bounds for sums of random variables

$\longrightarrow$ Kaas, Dhaene and Goovaerts (2000)

General result: Let U be a uniform(0,1) random variable. For any random vector $\vec{X} = (X_1, X_2, \ldots, X_n)$ with marginal cdf's $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$, we have

$$\sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F^{-1}_{X_i|\Lambda}(U) \leq_{cx} \sum_{i=1}^{n} F^{-1}_{X_i}(U)$$

which we denote by

$$\mathbb{S}^{\ell} \leq_{cx} \mathbb{S} \leq_{cx} \mathbb{S}^{u} \leq_{cx} \mathbb{S}^{c}$$

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$$\mathbb{S}^{\ell} \leq_{cx} \mathbb{S} \leq_{cx} \mathbb{S}^{u} \leq_{cx} \mathbb{S}^{c}$$

$\mathbb{S}^c$: keep marginals X_i but replace copula by most dangerous comonotonic copula

$\mathbb{S}^c$ is always a comonotonic sum

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$\mathbb{S}^{\ell}$: replace marginals X_i by less dangerous $\mathrm{E}[X_i|\Lambda]$

$\mathbb{S}^{\ell}$ is comonotonic sum if all $\mathrm{E}[X_i|\Lambda]$ are $\nearrow$ functions of Λ

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$F^{-1}_{X_i|\Lambda}(U)$ stands for the rv $f_i(U, \Lambda)$ with $f_i(u, \lambda) = F^{-1}_{X_i|\Lambda=\lambda}(u)$

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$$\Downarrow$$

$$\mathrm{E}[\left(\mathbb{S}^{\ell} - d\right)_+] \leq \mathrm{E}[(\mathbb{S} - d)_+] \leq \mathrm{E}[(\mathbb{S}^{u} - d)_+] \leq \mathrm{E}[(\mathbb{S}^{c} - d)_+] \;\forall\; d$$

(notation: $\pi^{\ell}(\mathbb{S}, d, \Lambda) \leq \pi(\mathbb{S}, d) \leq \pi^{u}(\mathbb{S}, d, \Lambda) \leq \pi^{c}(\mathbb{S}, d), \;\forall\; d$)

- Kaas et al. (2000):
  - the quantile function is additive for comonotonic risks
  - in case of strictly increasing and continuous marginals, the cdf $F_{\mathbb{S}^c}(x)$ is uniquely determined by

$$F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}\left(x\right)\right) = \sum_{i=1}^{n} F_{X_i}^{-1}\left(F_{\mathbb{S}^c}\left(x\right)\right) = x, \qquad \left(F_{\mathbb{S}^c}^{-1}(0) < x < F_{\mathbb{S}^c}^{-1}(1)\right)$$

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- Dhaene et al. (2002):
  stop-loss premiums of a sum of comonotonic random variables can easily be obtained from stop-loss premiums of the terms:

$$\pi^c(\mathbb{S}, d) = \sum_{i=1}^{n} \mathrm{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{\mathbb{S}^c}\left(d\right)\right)\right)_+\right], \qquad \left(F_{\mathbb{S}^c}^{-1}(0) < d < F_{\mathbb{S}^c}^{-1}(1)\right)$$

- lower bound:
  - $\mathbb{S}^\ell$ comonotonic:

$$\pi^\ell(\mathbb{S}, d, \Lambda) = \sum_{i=1}^{n} \mathrm{E}\left[\left(\mathrm{E}\left[X_i \mid \Lambda\right] - F^{-1}_{\mathrm{E}[X_i|\Lambda]}\left(F_{\mathbb{S}^\ell}(d)\right)\right)_+\right]$$

$$= \sum_{i=1}^{n} \mathrm{E}\left[\left(\mathrm{E}\left[X_i \mid \Lambda\right] - \mathrm{E}\left[X_i \mid \Lambda = F^{-1}_{\Lambda}\left(F_{\mathbb{S}^\ell}(d)\right)\right]\right)_+\right]$$

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$$\left(F^{-1}_{\mathbb{S}^\ell}(0) < d < F^{-1}_{\mathbb{S}^\ell}(1)\right)$$

  - $\mathbb{S}^\ell$ not comonotonic:

$$\pi^\ell(\mathbb{S}, d, \Lambda) = \int_{-\infty}^{+\infty} \left(\sum_{i=1}^{n} \mathrm{E}\left[X_i \mid \Lambda = \lambda\right] - d\right)_+ dF_\Lambda(\lambda)$$

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- improved upper bound:

$$\pi^u(\mathbb{S},d,\Lambda) = \int_{-\infty}^{+\infty} \sum_{i=1}^{n} \mathrm{E}\left[\left(X_i - F^{-1}_{X_i|\Lambda=\lambda}\left(F_{\mathbb{S}^u|\Lambda=\lambda}(d)\right)\right)_+ \mid \Lambda = \lambda\right] dF_\Lambda(\lambda)$$

$$\left(F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(0) < d < F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(1)\right)$$

Upper bounds based on the lower bound plus an error term

based on the ideas of Rogers and Shi (1995) :

$$0 \le \mathrm{E}\left[\mathrm{E}\left[Y_+ \mid Z\right] - \mathrm{E}\left[Y \mid Z\right]_+\right] \le \frac{1}{2}\mathrm{E}\left[\sqrt{\mathsf{Var}(Y \mid Z)}\right]$$

$Y := \mathbb{S} - d$ and $Z := \Lambda$

$$\Rightarrow \quad 0 \le \mathrm{E}\left[\mathrm{E}\left[(\mathbb{S} - d)_+ \mid \Lambda\right] - (\mathbb{S}^\ell - d)_+\right] \le \frac{1}{2}\mathrm{E}\left[\sqrt{\mathsf{Var}(\mathbb{S} \mid \Lambda)}\right]$$

$$\Downarrow$$

$$\pi^{eub}(\mathbb{S}, d, \Lambda) = \pi^\ell(\mathbb{S}, d, \Lambda) + \tfrac{1}{2}\mathrm{E}\left[\sqrt{\mathsf{Var}(\mathbb{S} \mid \Lambda)}\right]$$

$$\begin{aligned}
\mathrm{E}\left[\sqrt{\mathsf{Var}(\mathbb{S} \mid \Lambda)}\right] &= \mathrm{E}\left[\left(\mathrm{E}\left[\mathbb{S}^2 \mid \Lambda\right] - \left(\mathrm{E}\left[\mathbb{S} \mid \Lambda\right]\right)^2\right)^{1/2}\right] \\
&= \mathrm{E}\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}\left[X_i X_j \mid \Lambda\right] - \left(\mathbb{S}^\ell\right)^2\right)^{1/2}\right]
\end{aligned}$$

Decomposition of the stop-loss premium

if $\exists\ d_\Lambda$ such that $\Lambda \geq d_\Lambda \Rightarrow \mathbb{S} \geq d$ (or such that $\Lambda \leq d_\Lambda \Rightarrow \mathbb{S} \geq d$)

$$\Rightarrow \mathrm{E}[(\mathbb{S}-d)_+ \mid \Lambda] = \mathrm{E}[\mathbb{S}-d \mid \Lambda] = (\mathbb{S}^\ell - d)_+$$

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$$\begin{aligned} \pi(\mathbb{S},d) &= \int_{-\infty}^{d_\Lambda} \mathrm{E}[(\mathbb{S}-d)_+|\Lambda=\lambda]dF_\Lambda(\lambda) + \int_{d_\Lambda}^{+\infty} \mathrm{E}[\mathbb{S}-d|\Lambda=\lambda]dF_\Lambda(\lambda) \\ &\stackrel{not}{=} I_1 + I_2 \end{aligned}$$

$$I_2 = \int_{d_\Lambda}^{+\infty} \sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda=\lambda]dF_\Lambda(\lambda) - d(1-F_\Lambda(d_\Lambda))$$

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$$\begin{aligned} I_2 &= \int_{d_\Lambda}^{+\infty} \sum_{i=1}^{n} \mathrm{E}[X_i | \Lambda=\lambda] dF_\Lambda(\lambda) - d(1-F_\Lambda(d_\Lambda)) \\ I_1 &\leq I_1^{upp} := \int_{-\infty}^{d_\Lambda} \mathrm{E}[(\mathbb{S}^u - d)_+ \mid \Lambda=\lambda] dF_\Lambda(\lambda) \end{aligned}$$

partially exact/comonotonic upper bound: $\pi^{pecub}(\mathbb{S},d,\Lambda) = I_1^{upp} + I_2$

$$I_1 \;\geq\; I_1^{low} := \int_{-\infty}^{d_\Lambda} \left(\mathrm{E}[\mathbb{S} \mid \Lambda = \lambda] - d\right)_+ dF_\Lambda(\lambda)$$
$$= \int_{-\infty}^{d_\Lambda} \left(\sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda = \lambda] - d\right)_+ dF_\Lambda(\lambda)$$

upper bound based on lower bound dependent on retention

$$\begin{aligned}
0 \;&\leq\; I_1 - I_1^{low} \\
&= \int_{-\infty}^{d_\Lambda} \Big(\mathrm{E}[(\mathbb{S} - d)_+ \mid \Lambda = \lambda] - \left(\mathrm{E}\left[\mathbb{S} \mid \Lambda = \lambda\right] - d\right)_+\Big)\, dF_\Lambda(\lambda) \\
&\leq \frac{1}{2} \int_{-\infty}^{d_\Lambda} \left(\mathrm{Var}\left(\mathbb{S} \mid \Lambda = \lambda\right)\right)^{\frac{1}{2}} dF_\Lambda(\lambda) \qquad (1) \\
&\leq \frac{1}{2} \left(\mathrm{E}\left[\mathrm{Var}\left(\mathbb{S} \mid \Lambda\right) 1_{\{\Lambda < d_\Lambda\}}\right]\right)^{\frac{1}{2}} \left(\mathrm{E}\left[1_{\{\Lambda < d_\Lambda\}}\right]\right)^{\frac{1}{2}} \equiv \varepsilon(d_\Lambda)
\end{aligned}$$

$$\boxed{\pi^{deub}(\mathbb{S}, d, \Lambda) = \pi^{\ell}(\mathbb{S}, d, \Lambda) + \varepsilon(d_\Lambda)}$$

Note: $\pi^{deub}(\mathbb{S}, d, \Lambda) = \pi^{eub}(\mathbb{S}, d)$ for $d_\Lambda = +\infty$ in (1)

Application to life annuities with stochastic interest rates

stop-loss premium of a sum of lognormal random variables with a stochastic time horizon K_x : $\mathbb{S}_x = \sum_{i=1}^{K_x} \alpha_i e^{Z_i}$

$$\pi(\mathbb{S}_x, d) = \mathrm{E}\left[(\mathbb{S}_x - d)_+\right] = \mathrm{E}_{K_x}\left[\mathrm{E}\left[\left(\sum_{i=1}^{K_x} \alpha_i e^{Z_i} - d\right)_+ \mid K_x\right]\right]$$
$$= \sum_{k=1}^{\infty} \mathsf{Pr}[K_x = k]\mathrm{E}\left[\left(\sum_{i=1}^{k} \alpha_i e^{Z_i} - d\right)_+\right]$$
$$= \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \; \pi(\mathbb{S}_k, d),$$

with $\mathbb{S}_k = \sum_{i=1}^{k} \alpha_i e^{Z_i}$ (deterministic time horizon!)

$$\pi(\mathbb{S}_x, d) = \mathrm{E}\left[(\mathbb{S}_x - d)_+\right] = \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \; \pi(\mathbb{S}_k, d)$$

$$\begin{aligned}
\pi^{bound}(\mathbb{S}_x, d, (\mathbf{\Lambda})) &= \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \pi^{bound}(\mathbb{S}_k, d, (\Lambda_k)) \\
\pi^{emub}(\mathbb{S}_x, d, \mathbf{\Lambda}) &= \pi^{\ell}(\mathbb{S}_x, d, \mathbf{\Lambda}) + \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \min\left(\frac{1}{2}\mathrm{E}\left[\sqrt{\mathsf{Var}(\mathbb{S}_k|\Lambda_k)}\right], \varepsilon(d_{\Lambda_k})\right) \\
\pi^{min}(\mathbb{S}, d, \mathbf{\Lambda}) &= \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \min\left(\pi^{c}(\mathbb{S}_k, d), \pi^{u}(\mathbb{S}_k, d, \Lambda_k), \pi^{pecub}(\mathbb{S}_k, d, \Lambda_k), \pi^{emub}(\mathbb{S}_k, d, \Lambda_k)\right)
\end{aligned}$$

- **Lemma**: Let X be a lognormal random variable of the form αe^Z with $Z \sim N(\mathrm{E}[Z], \sigma_Z)$ and $\alpha \in \mathbb{R}$. Then the stop-loss premium with retention d equals for $\alpha d > 0$

$$\boxed{\pi(X, d) = \mathrm{sign}(\alpha) e^{\mu + \frac{\sigma^2}{2}} \Phi(\mathrm{sign}(\alpha) b_1) - d\Phi(\mathrm{sign}(\alpha) b_2),}$$

where

$$\mu = \ln|\alpha| + \mathrm{E}[Z] \qquad \sigma = \sigma_Z$$

$$b_1 = \frac{\mu + \sigma^2 - \ln|d|}{\sigma} \qquad b_2 = b_1 - \sigma$$

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- comonotonic upper bound:

$$\boxed{\pi^c(\mathbb{S}_k, d) = \sum_{i=1}^{k} \alpha_i\, e^{\mathrm{E}[Z_i] + \frac{\sigma_{Z_i}^2}{2}}\, \Phi\left[\operatorname{sign}(\alpha_i)\sigma_{Z_i} - \Phi^{-1}(F_{\mathbb{S}_k^c}(d))\right] - d\left(1 - F_{\mathbb{S}_k^c}(d)\right)}$$

where $F_{\mathbb{S}_k^c}(d)$ can be obtained by solving:

$$\sum_{i=1}^{k} \alpha_i\, e^{\mathrm{E}[Z_i] + \operatorname{sign}(\alpha_i)\sigma_{Z_i}\Phi^{-1}(F_{\mathbb{S}_k^c}(x))} = x$$

Choice of the conditioning and decomposition variable

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(a) 'Taylor-based': $\gamma_i = \alpha_i e^{\mathrm{E}[Z_i]}$

$\Lambda_k \rightarrow$ linear transformation of first order approximation to $\mathbb{S}_k$

$$\Lambda_k \geq d_{\Lambda_k} \implies \mathbb{S}_k \geq d$$

$$d_{\Lambda_k} = d - \sum_{i=1}^{k} \alpha_i e^{\mathrm{E}[Z_i]} \left(1 - \mathrm{E}[Z_i]\right)$$

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(b) 'Maximum variance': $\gamma_i = \alpha_i \mathrm{E}\left[e^{Z_i}\right] = \alpha_i e^{\mathrm{E}[Z_i] + \frac{1}{2}\sigma^2_{Z_i}}$

maximization of first order approximation for Var$(\mathbb{S}_k^\ell)$:

$$\text{Var}(\mathbb{S}_k^\ell) \approx \left(\text{Corr}[\sum_{i=1}^{k} \alpha_i \mathrm{E}[e^{Z_i}] Z_i, \Lambda_k]\right)^2 \text{Var}[\sum_{i=1}^{k} \alpha_i \mathrm{E}[e^{Z_i}] Z_i]$$

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$$\Lambda_k \geq d_{\Lambda_k} \implies \mathbb{S}_k \geq d \qquad \boxed{d_{\Lambda_k} = d - \sum_{i=1}^{k} \alpha_i \mathrm{E}[e^{Z_i}] \left(1 - \mathrm{E}[Z_i] - \frac{1}{2}\sigma_{Z_i}^2\right)}$$

Choice of the conditioning and decomposition variable

- conditioning random variable: $\Lambda_k = \sum_{i=1}^{k} \gamma_i Z_i$
- based on the standardized logarithm of the geometric average $\mathbb{G} = (\prod_{i=1}^{k} \mathbb{S}_k)^{1/k}$ as in Nielsen and Sandman (2002)

$$\Lambda_k = \frac{\ln \mathbb{G} - \mathrm{E}[\ln \mathbb{G}]}{\sqrt{\mathsf{Var}[\ln \mathbb{G}]}} = \frac{\sum_{i=1}^{k}(Z_i - \mathrm{E}[Z_i])}{\sqrt{\mathsf{Var}(\sum_{i=1}^{k} Z_i)}}$$

$$\Lambda_k \geq d_{\Lambda_k} \implies \mathbb{S}_k \geq d$$

$$d_{\Lambda_k} = \frac{k \ln\left(\frac{d}{k}\right) - \sum_{i=1}^{k}(\mathrm{E}[Z_i] + \ln(\alpha_i))}{\sqrt{\mathsf{Var}(\sum_{i=1}^{k} Z_i)}}$$

Numerical illustration

- mortality process: Makeham's model

  the number of persons alive at age x

  $$l_x = a s^x g^{c^x}$$

  ($a > 0, 0 < s < 1, 0 < g < 1$ and $c > 1$) with parameters ($l_0 = 1000000$)
  for men: $a = 1000266.63$, $s = 0.999441703848$, $g = 0.999733441115$,
  $c = 1.101077536030$

- discounting process: Brownian motion

  $$e^{Z_i} = e^{-Y(i)} := e^{-(Y_1+\cdots+Y_i)}$$

  yearly returns Y_i are i.i.d. normally distributed with mean $\mu = 0.07$ and volatility $\sigma = 0.1$

  $$\Rightarrow \quad Y(i) \sim 0.07i + 0.1B(i)$$

- consider a whole life annuity on a life (65) (male) with yearly unit payments

$$\mathrm{E}\left[\left(\mathbb{S}_{65}^{policy}-d\right)_+\right]=\sum_{k=1}^{\lfloor\omega-65\rfloor} {}_{k|}q_{65}\mathrm{E}\left[\left(\tilde{\mathbb{S}}_k-d\right)_+\right]$$

$\hookrightarrow \tilde{\mathbb{S}}_k=\mathbb{S}_k$ with $\alpha_i=1$ and $Z_i=-Y(i)$ $(i=1,\ldots,k)$

Monte Carlo estimates based on 50 $\times$ 1 000 000 simulations

d	LB	MC (s.e. $\times 10^5$)	MIN	EMUB	PECUB	ICUB	CUB
0	**9.3196**	**9.3196**	**9.3196**	9.3196	9.3196	9.3196	9.3196
5	**4.6191**	**4.6191** (8.49)	**4.6195**	4.6197	4.6219	4.6238	4.6244
10	**1.2269**	**1.2304** (5.48)	**1.2385**	1.2400	1.2839	1.3277	1.3389
15	**0.1737**	**0.1739** (0.51)	**0.2070**	0.2145	0.2381	0.2530	0.2610
20	**0.0207**	**0.0216** (0.19)	**0.0444**	0.0718	0.0451	0.0454	0.0480
25	**0.0026**	**0.0026** (0.01)	**0.0088**	0.0545	0.0088	0.0088	0.0095
30	**0.0004**	**0.0004** (0.002)	**0.0019**	0.0522	0.0019	0.0019	0.0021

- consider a portfolio of N_0 homogeneous life annuity contracts for which the future lifetimes of the insureds $T_x^{(1)}, T_x^{(2)}, \ldots, T_x^{(N_0)}$ are assumed to be independent

  $\Rightarrow$ for sufficiently large N_0 (Law of Large Numbers)

$$\mathrm{E}\left[\left(\sum_{i=1}^{\lfloor\omega-x\rfloor} N_i e^{-Y(i)} - d\right)_+\right] = \mathrm{E}\left[N_0\left(\sum_{i=1}^{\lfloor\omega-x\rfloor} \frac{N_i}{N_0} e^{-Y(i)} - \frac{d}{N_0}\right)_+\right]$$

$$\approx N_0 \mathrm{E}\left[\left(\sum_{i=1}^{\lfloor\omega-x\rfloor} {}_ip_x e^{-Y(i)} - \frac{d}{N_0}\right)_+\right],$$

  where N_i denotes the number of survivals after the i-th year

  $\Rightarrow$ for large portfolios of life annuities it suffices to compute stop-loss premiums of an "average" portfolio $\mathbb{S}_x^{average}$:

$$\mathbb{S}_x^{average} = \sum_{i=1}^{\lfloor\omega-x\rfloor} {}_ip_x e^{-Y(i)}$$

  $\hookrightarrow \alpha_i = {}_ip_x \quad (i = 1, \ldots, \lfloor\omega - x\rfloor)$

consider $\mathbb{S}_{65}^{average}$ (male persons)

Monte Carlo estimates based on 50 $\times$ 1 000 000 simulations

d	LB	MC (s.e. $\times 10^5$)	EMUB	PECUB	ICUB	CUB
0	**9.3196**	**9.3196**	**9.3196**	9.3196	9.3196	9.3196
5	**4.3200**	**4.3200** (0.37)	**4.3202**	4.3219	4.3227	4.3233
10	**0.5533**	**0.5543** (0.13)	**0.5784**	0.6557	0.7081	0.7217
15	**0.0193**	**0.0197** (0.035)	0.0744	**0.0524**	0.0524	0.0559

Remark:

- $\mathrm{E}[\mathbb{S}_x^{average}] = \mathrm{E}[\mathbb{S}_x^{policy}]$
- $\mathbb{S}_x^{average} \leq_{cx} \mathbb{S}_x^{policy} \Rightarrow \mathrm{E}[(\mathbb{S}_x^{average} - d)_+] \leq \mathrm{E}[(\mathbb{S}_x^{policy} - d)_+] \; \forall d > 0$

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- methodology of estimating stop-loss premiums of strongly dependent random variables
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  decompose value of stop-loss premium by conditioning
  apply best (smallest) upper bound on each component separately
- numerical illustrations: LB very accurate
  decomposition significantly improves UB

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