Integrated risk management when stock prices follow an exponential Lévy process

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Outline:

(1) The model.

1.1 Insurance model.

1.2 Investment model.

1.3 Integrated model.

(2) Risk measurement.

2.1 Stationarity conditions.

2.2 Tail behaviour, regularly varying claims.

2.3 Tail behaviour, claims with finite moments.

(3) Summary, examples and applications.

Insurance model:

$$U(t) = u + ct - S(t)\,, \quad t \geq 0\,,$$

- $u > 0$: initial capital;

- $c > 0$: premium rate;

- $S(t) = \sum_{j=1}^{N(t)} Y_j\,, \quad t \geq 0$: total claim amount process;

- $\{Y_j\}_{j=1}^{\infty}$ - iid rv's : the claim sizes with df F;

- $(N(t))_{t\geq 0}$ - Poisson process (λ) with claim arrival times T_j, $j = 1, 2, \ldots$;

- $\{Y_j\}_{j=1}^{\infty} \perp (N(t))_{t\geq 0}$.

Riskless asset:

$$X_0(t) = e^{\delta t}\,, \quad t \geq 0,$$

$\delta > 0$ - constant interest rate.

Risky asset:

$$X_1(t) = \exp\left(L(t)\right)\,, \quad t \geq 0,$$

$(L(t))_{t\geq 0}$ is a Lévy process with triplet (γ, σ^2, ν), $\gamma, \sigma > 0$, ν - Lévy measure.

Assumption: $E[X_1(1)] > X_0(1)$.

The investment process: for $\theta \in [0,1]$ define

$$dX_\theta(t) = X_\theta(t-)\left((1-\theta)\delta dt + \theta d\widehat{L}(t)\right), \quad t > 0, \; X_\theta(0) = 1.$$

where $\mathcal{E}(\widehat{L}) = \exp(L)$.

The SDE has the solution (Emmer and Klüppelberg (2004))

$$X_\theta(t) = \exp(L_\theta(t)),$$

where $(L_\theta(t))_{t\geq 0}$ - Lévy process with triplet $(\gamma_\theta, \sigma_\theta, \nu_\theta)$.

Transformation for the jumps:

$$\Delta L_\theta = \log(1 + \theta(e^{\Delta L} - 1)) \geq \log(1-\theta).$$

Goals:

(1) Combine insurance and investment in an integrated risk model.

(2) Quantify the risk in a stationary way.

(3) Investigate the stationary distribution and its tail.

(4) Obtain explicit results w.r.t. the investment strategy θ.

Integrated risk process: for $\theta \in [0,1]$ define

$$dU_\theta(t) = c\,dt - dS(t) + U_\theta(t-)\left((1-\theta)\delta dt + \theta d\widehat{L}(t)\right), \quad t > 0\,,\ U(0) = u\,.$$

where $\mathcal{E}(\widehat{L}) = \exp(L)$.

If S is independent of L, then the SDE has the solution

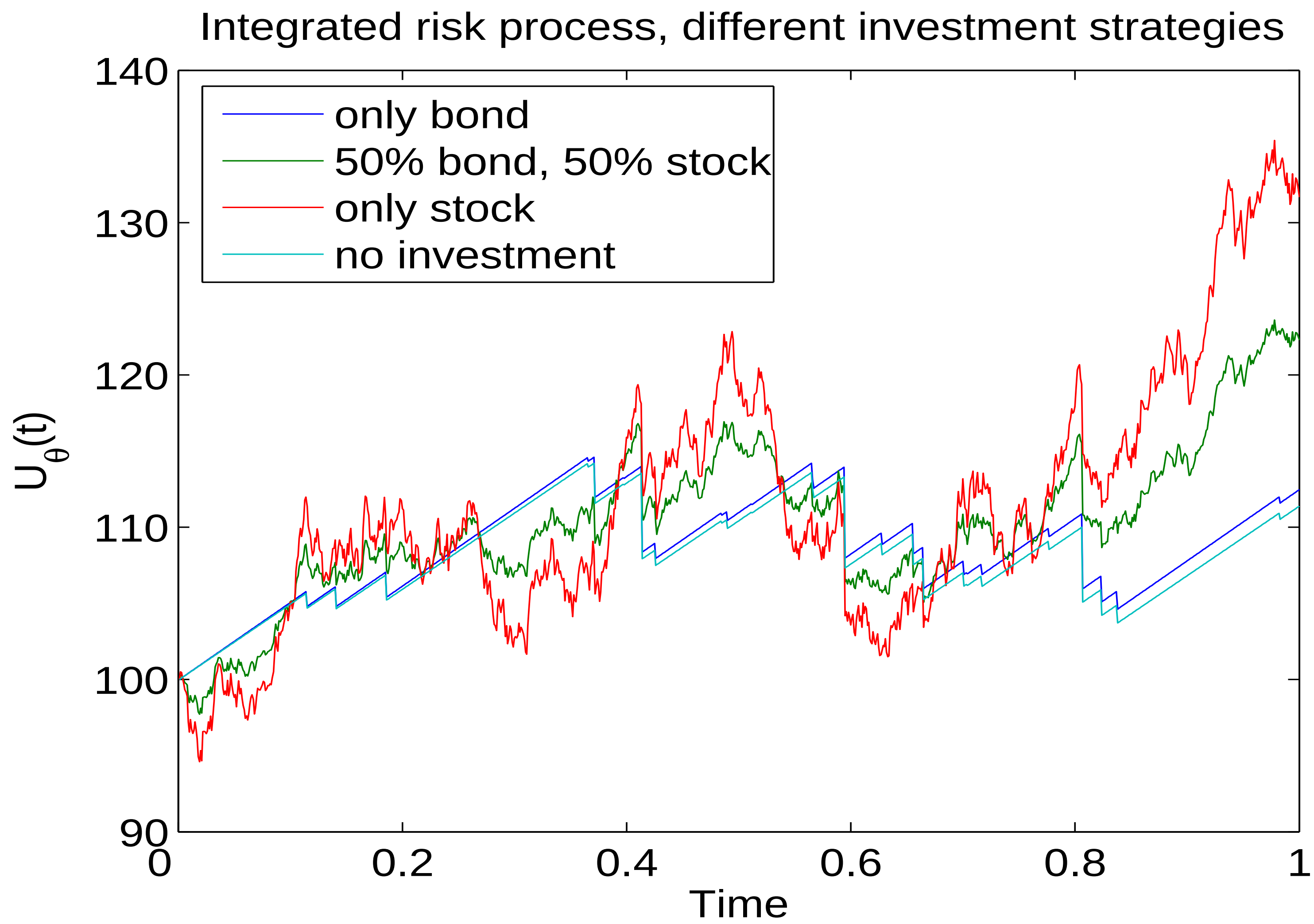
$$U_\theta(t) = \exp(L_\theta(t))\left(u + \int_0^t \exp(-L_\theta(v))\,(c\,dv - dS(v))\right),\ t \geq 0\,.$$

Recognize $(U_\theta(t))_{t\geq 0}$ as a generalised Ornstein-Uhlenbeck process, see e.g. Lindner and Maller (2004).

Simulated sample paths

Parameters: $u = 100$, $c = 50$, $\lambda = 20$, $Y \stackrel{d}{=} Exp(0.5)$, $T = 1$, $\delta = 0.01$,

L - Brownian motion with $\gamma = 0.04$ and $\sigma = 0.2$.



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Discounted net loss process:

$$V_\theta(t) = u - e^{-L_\theta(t)} U_\theta(t) = \int_0^t \exp(-L_\theta(v))\,(dS(v) - c\,dv)\,, \;\; t \geq 0\,.$$

Note:

$$P(U_\theta(t) < 0 \,|\, U_\theta(0) = u) = P(V_\theta(t) > u), \; t > 0\,.$$

Lindner and Maller (2004) give NASC for

$$V_\theta(t) \overset{\text{a.s.}}{\to} V_\theta^\infty, \; t \to \infty\,,$$

for some finite rv V_θ^∞.

Discrete skeleton: Recall T_k – time of the k-th claim. Define for $k \in \mathbb{N}$

$$\begin{aligned} A_\theta^k &= \int_{T_{k-1}}^{T_k} \exp(-(L_\theta(v) - L_\theta(T_{k-1})))\,(dS(v) - cdv)\,, \\ B_\theta^k &= \exp(-(L_\theta(T_k) - L_\theta(T_{k-1}))\,. \end{aligned}$$

Then $(A_\theta^k, B_\theta^k)_{k\in\mathbb{N}}$ is a sequence of iid random vectors.

Define $(V_\theta^k)_{k\in\mathbb{N}}$ by the backward stochastic recurrence equation (BSRE):

$$V_\theta^k = \sum_{m=1}^{k} A_\theta^m \prod_{j=1}^{m-1} B_\theta^j\,,\ \ k \in \mathbb{N}\,.$$

Then $V_\theta(T_k) = V_\theta^k\,,\ \ k \in \mathbb{N}\,.$

Stationarity:

Denote $\varphi_\theta(s) := \Psi_\theta(is) = \log E\left[\exp(-sL_\theta(1))\right]\,,\ s \geq 0$.

If $\varphi_\theta(1) < \lambda$, $E[Y] < \infty$ and $E[L_\theta(1)] > 0$ then

$$V_\theta^k \stackrel{\text{a.s.}}{\to} V_\theta^\infty\,,\ k \to \infty\,, \quad \text{and} \quad V_\theta(t) \stackrel{\text{a.s.}}{\to}\ V_\theta^\infty\,,\ t \to \infty\,,$$

for some finite rv V_θ^∞.

Furthermore, $V_\theta^\infty \perp (A_\theta,\, B_\theta)$ and

$$V_\theta^\infty \stackrel{d}{=} A_\theta + B_\theta V_\theta^\infty\,.$$

Lemma 1 Let $\theta \in (0, 1]$ and denote by $\mathcal{V}_\theta = \{v \geq 0 : \varphi_\theta(v) < \infty\}$. Assume

(a) $\sup \mathcal{V}_\theta \notin \mathcal{V}_\theta$;

(b) $E[L_\theta(1)] > 0$;

(c) $P(L_\theta(1) < 0) > 0$.

Then there exists a unique, positive $\kappa(\theta) \in int\mathcal{V}_\theta$, such that

$$\varphi_\theta(\kappa(\theta)) = 0\,.$$

Remarks:

(a) If $\theta \in [0, 1)$, then $\mathcal{V}_\theta = \mathbb{R}^+$.

(b) If $E[L(1)] > 0$, then $E[L_\theta(1)] > 0$, $\theta \in [0, 1]$

(c) If $\sigma > 0$ or $\nu((-\infty, 0)) > 0$, then $P(L_\theta(1) < 0) > 0$, $\theta \in (0, 1]$.

Pareto claims ($E[Y^{\kappa(\theta)}] = \infty$):

Let $V_\theta^\infty \stackrel{d}{=} A_\theta + B_\theta V_\theta^\infty$ and the conditions of Lemma 1 hold. Let also for some $\alpha < \kappa(\theta)$

$$P(Y > x) \sim c_Y x^{-\alpha}, \;\; x \to \infty.$$

Then, applying Konstantinides and Mikosch (2004),

$$P(V_\theta^\infty > x) \sim \frac{\lambda c_Y}{|\varphi_\theta(\alpha)|} x^{-\alpha}, \;\; x \to \infty.$$

The case $E[Y^{\kappa(\theta)}] < \infty$:

Let $V_\theta^\infty \stackrel{d}{=} A_\theta + B_\theta V_\theta^\infty$ and the conditions of Lemma 1 hold. Let also

$$E[Y^{\kappa(\theta)}] < \infty\,.$$

Then, applying Goldie (1991),

$$P(V_\theta^\infty > x) \sim C_+(\theta) x^{-\kappa(\theta)}\,,\;\; P(V_\theta^\infty < -x) \sim C_-(\theta) x^{-\kappa(\theta)},\;\; x \to \infty\,,$$

where

$$C_{+,-}(\theta) = \frac{E\left[\left(\left(A_\theta + B_\theta V_\theta^\infty\right)^{+,-}\right)^{\kappa(\theta)} - \left(\left(B_\theta V_\theta^\infty\right)^{+,-}\right)^{\kappa(\theta)}\right]}{\kappa(\theta) m(\theta)}\,,$$

with $m(\theta) = E\left[B_\theta^{\kappa(\theta)} \log B_\theta\right] \in (0,\infty)$. Moreover, $C_-(\theta) + C_+(\theta) > 0\,.$

Under the additional conditions:

- Y has right unbounded support;

- $E[Y^{\kappa(\theta)+\beta}] < \infty$ for some $\beta > 0$ such that $\varphi_\theta(\kappa(\theta) + \beta) < \lambda$;

there exist $C_+(\theta) > 0$ and $C_-(\theta) > 0$ such that for $x \to \infty$

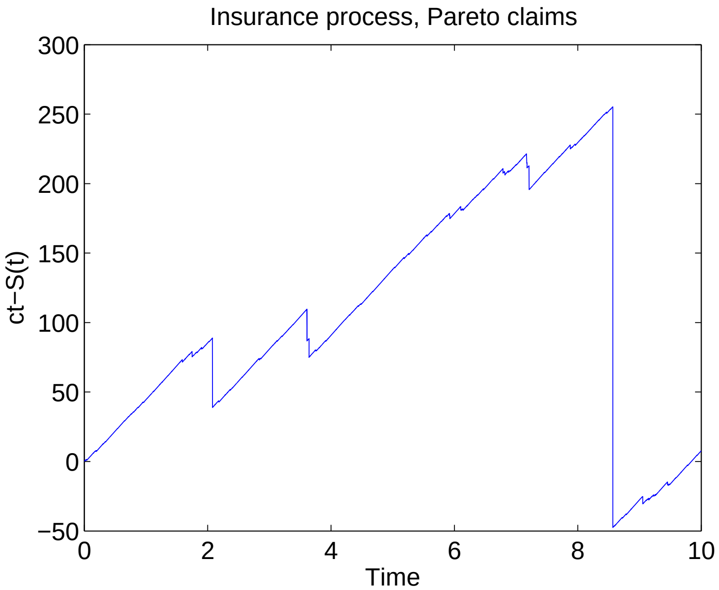
$$P(V_\theta^\infty > x) = C_+(\theta) x^{-\kappa(\theta)} + O(x^{-(\kappa(\theta)+\beta/2)}),$$

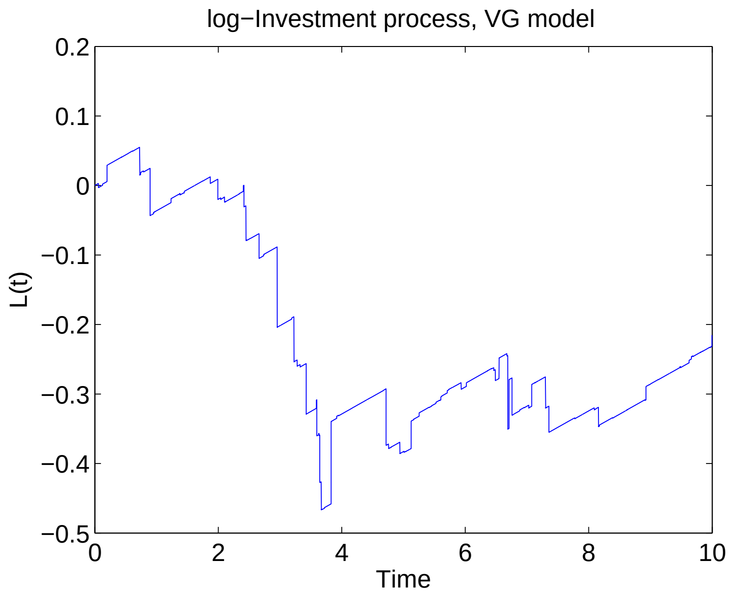
$$P(V_\theta^\infty < -x) = C_-(\theta) x^{-\kappa(\theta)} + O(x^{-(\kappa(\theta)+\beta/2)}).$$

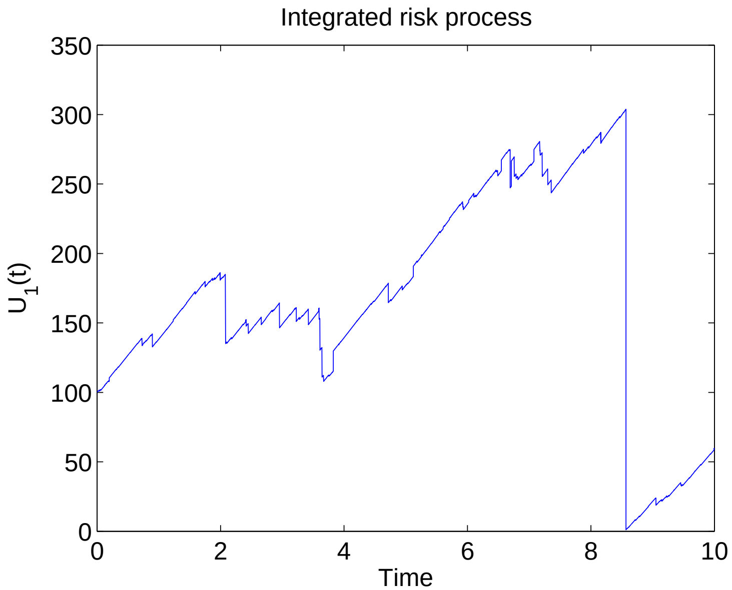
Summary, examples and applications.

	Dangerous claims	Dangerous investment
Claims	$\overline{F}(x) \sim c_Y x^{-\alpha}, \ \alpha < \kappa(\theta)$	$E[Y^{\kappa(\theta)+\beta}] < \infty, \ \beta > 0$
Tail behaviour	$P(V_\theta^\infty > x) \sim C(\theta) x^{-\alpha}$	$P(V_\theta^\infty > x) \sim C_+(\theta) x^{-\kappa(\theta)}$
Sensitivity w.r.t. θ	Pareto constant	Pareto exponent
Optimization w.r.t. θ	$C(\theta)$ is convex function	$\kappa(\theta)$ is decreasing function

Parameters: $u = 100$, $c = 50$, $\lambda = 20$, $\overline{F}(x) = (\frac{0.2}{0.2+x})^{-\alpha}$, $\alpha = 1.1$, $T = 10$, $\delta = 0.01$, $\theta = 1$, $L(t) = qt + W_{a,b}(S_\Gamma(t))$ - VG process with $q = 0.05$, $a = -0.01$, $b^2 = 0.04 - a^2$, var[S_Γ] $= 1$.

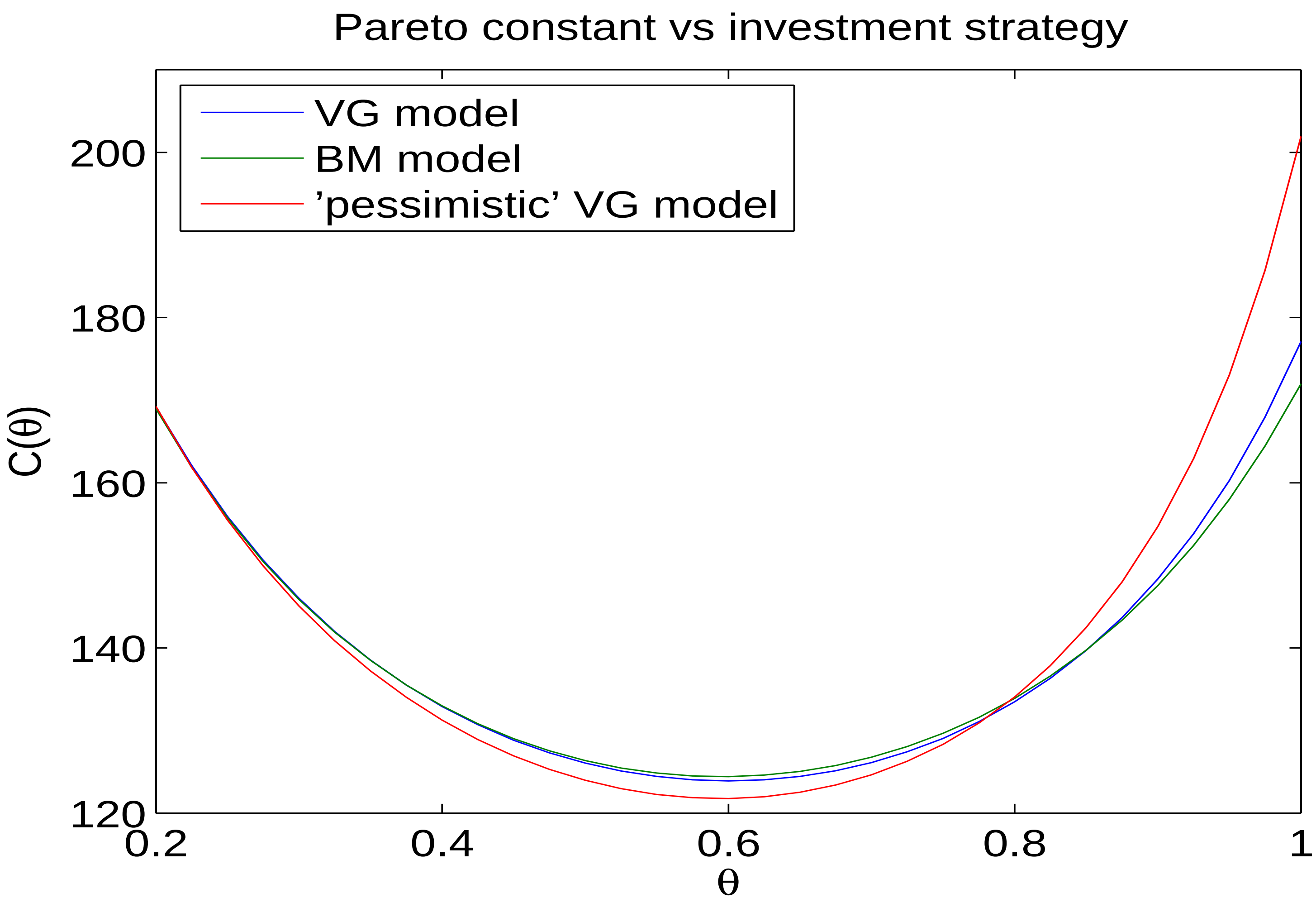






Analytic results, Pareto claims

If $\overline{F}(x) = (\frac{l}{l+x})^{-\alpha}$, $\alpha < \kappa(\theta)$, then $P(V_\theta^\infty > x) \sim C(\theta)x^{-\alpha}$, $C(\theta) = \lambda c_Y / |\varphi_\theta(\alpha)|$



$L(t) = qt + W_{a,b}(S_\Gamma(t))$ - "pessimistic" VG process with $q = 0.14$, $a = -0.1$, $b^2 = 0.04 - a^2$.

Parameters: $u = 100$, $c = 50$, $\lambda = 20$, $Y \stackrel{d}{=} Exp(0.5)$, $T = 10$, $\delta = 0.01$, $\theta = 1$,
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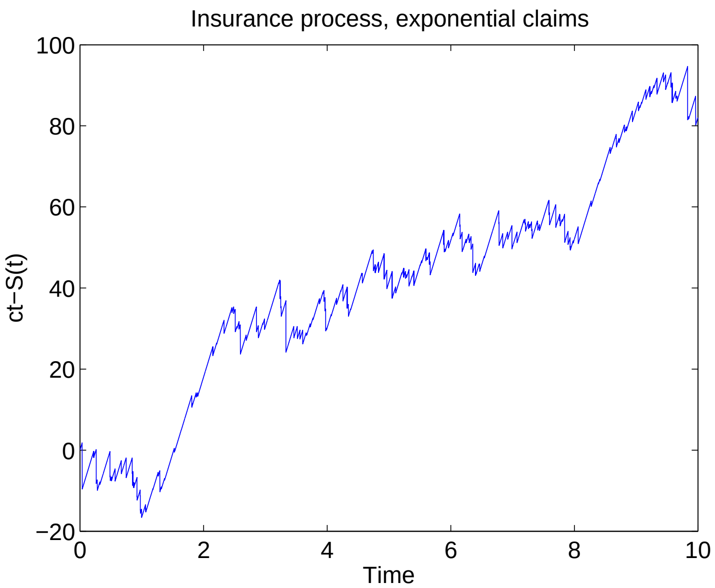
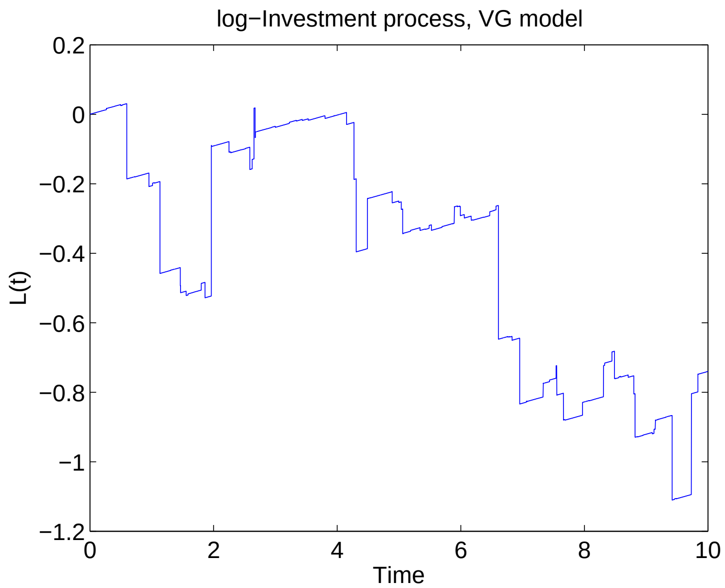
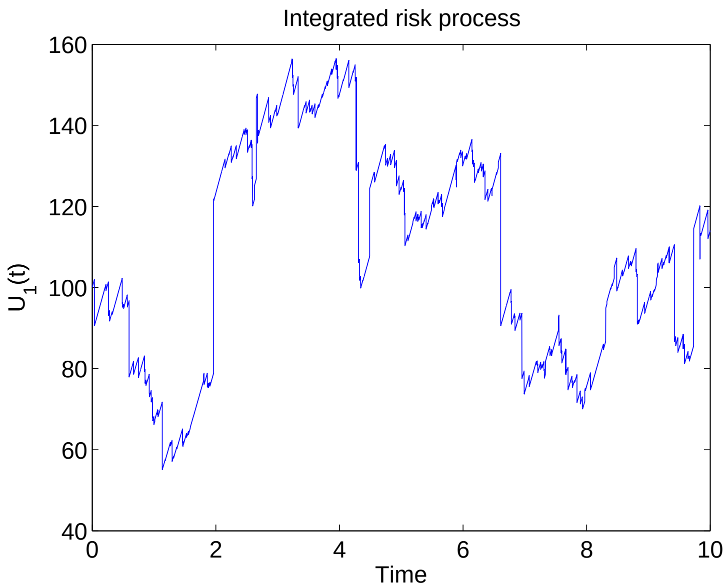
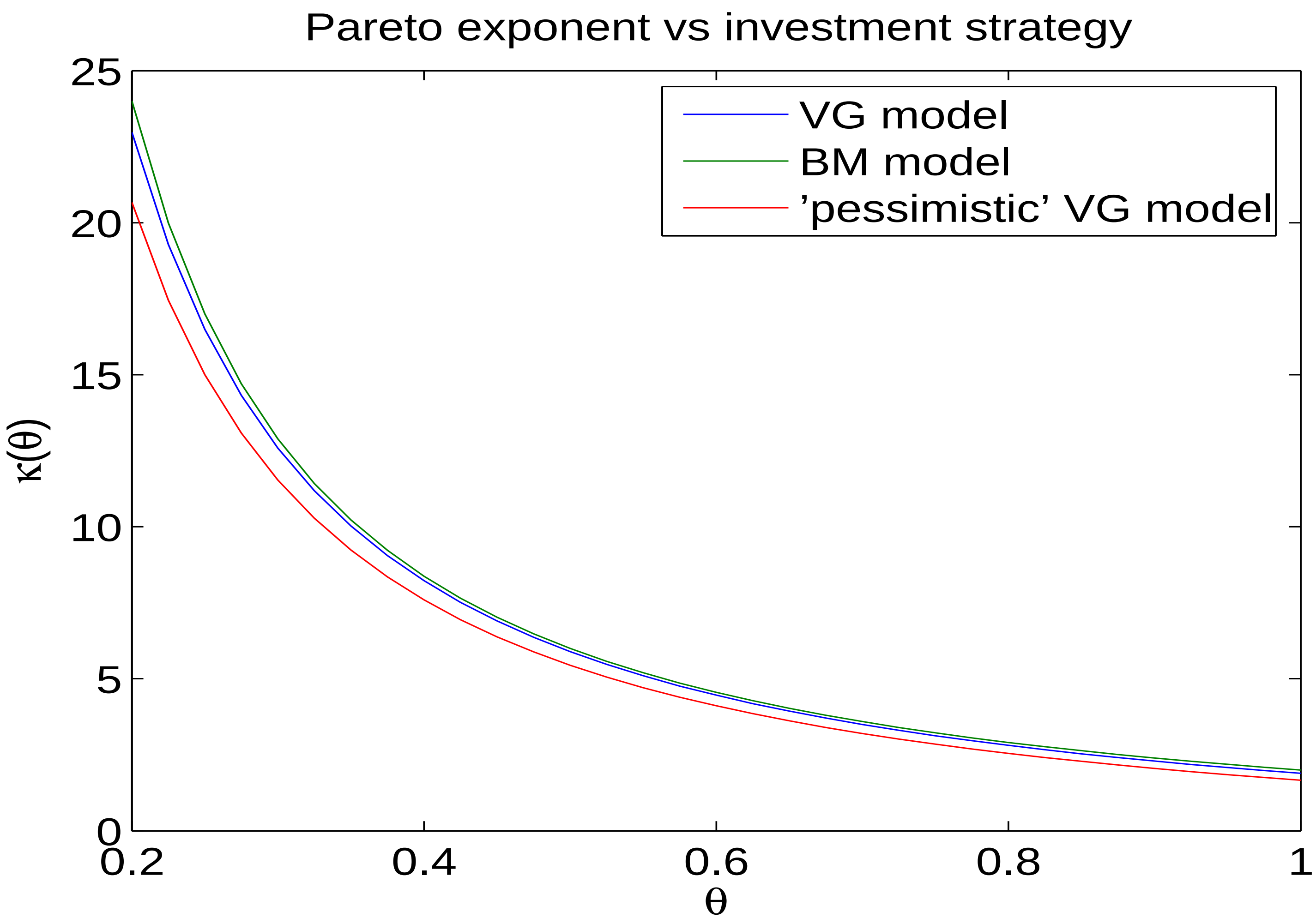






Analytic results, exponential claims

If $E[Y^{\kappa(\theta)+\beta}] < \infty$, $\beta > 0$, then $P(V_\theta^\infty > x) \sim C_+(\theta)x^{-\kappa(\theta)}$, $\varphi_\theta(\kappa(\theta)) = 0$.



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Thank you for your attention!