Multivariate Risk Processes in Insurance

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Reference:

Bregman, Y. and Klüppelberg, C. (2005)
Ruin estimates in multivariate models with Clayton dependence structure.
In preparation.

The Lévy insurance risk model:

$$R(t) = u + ct - X(t), \quad t \geq 0,$$

where $(X(t))_{t\geq 0}$ is a subordinator without drift.
Lévy-Khintchine representation: $Ee^{-\theta X(t)} = e^{t\Phi(\theta)}$ for $\theta \geq 0$ and

$$\Phi(\theta) = \int_{\mathbb{R}} (1 - e^{-\theta x})\Pi_X(dx)\,.$$

X has Lévy measure Π_X.

Assumptions:

- X has increasing sample paths: it has only upwards jumps.

Then $\Pi_X(-\infty, 0] = 0$ and $\int_{[0,1)} x\Pi_X(dx) < \infty$.

- R has positive drift: $EX(1)/c < 1$

Ruin probability (Pollacek-Khinchine formula)

$$
\begin{aligned}
\psi(u) &= P\left(R(t) < 0 \text{ for some } t > 0\right) \\
&= P\left(X(t) - ct > u \text{ for some } t > 0\right) \\
&= (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{H^{*n}}(u) \quad u \geq 0
\end{aligned}
$$

where $H(x) := \frac{1}{\mu}\int_0^x \overline{\Pi}_X(y)dy$ for $x \geq 0$, and $\overline{\Pi}_X(y) = \Pi_X([y,\infty))$, $y \geq 0$, is called **tail-integral**.

$\mu = EX(1) = \int_0^\infty \overline{\Pi}_X(y)dy$, $\overline{H} = 1 - H$, $\rho = \mu/c < 1$.

Example X compound Poisson $\Longrightarrow$ $\overline{\Pi}(x) = \lambda\overline{F}(x)$, $\mu = \lambda EY$. $\square$

Insurance portfolios

We consider the risk process of a portfolio consisting of d components

$$\begin{aligned} R(t) &= R_1(t) + \cdots + R_d(t) \\ &= u_1 + \cdots + u_d - (c_1 + \cdots + c_d)\, t - (X_1(t) + \cdots + X_d(t)) \quad t \geq 0 \end{aligned}$$

where $(X_1, \ldots, X_d)$ is a d-dimensional subordinator.

First note that $X(t) = X_1(t) + \cdots + X_d(t)$, $t \geq 0$, is also a subordinator.

Ruin probability (Pollacek-Khinchine formula)

$$\begin{aligned}\psi(u) &= P\left(R(t) < 0 \text{ for some } t > 0\right) \\ &= P\left(X(t) - ct > u \text{ for some } t > 0\right) \\ &= (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{H^{*n}}(u) \quad u \geq 0\end{aligned}$$

where $u = u_1 + \cdots + u_d$, $c = c_1 + \cdots + c_d$,

$H(x) := \frac{1}{\mu}\int_0^x \overline{\Pi}^+(y)dy$ for $x \geq 0$,

$\overline{\Pi}^+(y) = \overline{\Pi}_{X_1+\cdots+X_d}(y)$, $\mu = EX(1) = \int_0^\infty \overline{\Pi}^+(y)dy$, $\rho = \mu/c$.

Question

What is the influence of the dependence on the ruin probability?

Modelling dependence of Lévy processes Invoking the copula idea:

Definition [Copula]
A d-dimensional copula C is a distribution function on $[0,1]^d$ with standard uniform marginals. □

Properties $C : [0,1]^d \to [0,1]$ satisfies

(i) $C(u_1, \dots, u_d)$ is increasing in each component.
(ii) $C(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d)$
(iii) $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$ for all $i \in \{1, \dots, d\}$, $u_i \in [0,1]$.
(iv) For all $(a_1, \dots, a_d), (b_1, \dots, b_d) \in [0,1]^d$ with $a_i \leq b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(u_{1i_1}, \dots, u_{di_d}) \geq 0$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, \dots, d\}$.

Theorem [Sklar's Theorem]
Let F be a joint distribution function with marginals $F_1, \ldots, F_d$. Then there exists a copula $C : [0,1]^d \to [0,1]$ such that for all $x_1, \ldots, x_d \in \overline{\mathbb{R}} = [-\infty, \infty]$

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)). \quad (1)$$

If the marginals are continuous, then C is unique. Conversely, if C is a copula and $F_1, \ldots, F_d$ are distribution functions, then the function F as defined in (1) is a joint distribution function with marginals $F_1, \ldots, F_d$. $\square$

Question Can we use copulas to model dependence of Lévy processes?

Problems

The law of a Lévy process X is completely determined by the distribution of X at time t for some $t > 0$.

The copula C_t of $(X_1(t), \ldots, X_d(t))$ may depend on t. In general, C_s cannot be calculated from C_t, because C_s depends also on the marginal distributions.

For given infinitely divisible marginal distributions it is unclear, which copulas C_t yield multivariate infinite divisible distributions. (Copulas are invariant under strictly increasing transformations, infinite divisibility is not!)

Introduce Lévy copula

[Tankov (2003), Kallsen & Tankov (2004), Barndorff-Nielsen & Lindner (2004)].

For a subordinator, the Lévy measure plays the same role as the probability measure for random variables.

Problem Lévy measures may have a non-integrable singularity at 0.

Remedy Define a copula for the tail integral.

Definition Let Π be a Lévy measure on $[0,\infty)^d$.

Its **tail integral** $\overline{\Pi} : [0,\infty]^d \to [0,\infty]$ is such that

(1) $\overline{\Pi}(x_1,\dots,x_d) = \begin{cases} \Pi([x_1,\infty)\times\cdots\times[x_d,\infty)) & \text{for } x \in [0,\infty]^d \setminus \{0\}, \\ \infty & \text{for } x = 0; \end{cases}$

(2) $\overline{\Pi}$ is equal to 0, if one of its arguments is ∞;

(3) $\overline{\Pi}(0,\dots,x_i,\dots,0) = \overline{\Pi}_i(x_i)$, where $\overline{\Pi}_i(x_i) = \Pi([x_i,\infty))$. □

Definition [Subordinator copula/ S-copula]

A d-dimensional S-copula is a measure defining function $S : [0,\infty]^d \to [0,\infty]$ with marginals, which are the identity functions on $[0,\infty]$. $\square$

Properties Let $S : [0,\infty]^d \to [0,\infty]$ have domain $D_1 \times \cdots \times D_d$ and denote $\ell_i = \min D_i$, $u_i = \max D_i$ for $i = 1,\ldots,d$.

(i) $S(y_1,\ldots,y_d)$ is increasing in each component

(ii) $S(y_1,\ldots,y_{i-1},\ell_i,y_{i+1}\ldots,y_d) = 0$ for all $i \in \{1,\ldots,d\}$, $y_i \in D_i$.

(iii) $S(u_1,\ldots,u_{i-1},y_i,u_{i+1}\ldots,u_d) = y_i$ for all $i \in \{1,\ldots,d\}$, $y_i \in D_i$.

(iv) For all $(a_1,\ldots,a_d),(b_1,\ldots,b_d) \in D_1 \times \cdots \times D_d$ with $a_i \le b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} S(y_{1i_1},\ldots,y_{di_d}) \ge 0$$

where $y_{j1} = a_j$ and $y_{j2} = b_j$ for all $j \in \{1,\ldots,d\}$.

Theorem [Sklar's Theorem for S-copulae]
Let $\overline{\Pi}$ be the tail integral of a d-dimensional subordinator with marginal tail integrals $\overline{\Pi}_1, \ldots, \overline{\Pi}_d$. Then there exists a Lévy copula $S : [0,\infty]^d \to [0,\infty]$ such that for all $y_1, \ldots, y_d \in [0,\infty]$

$$\overline{\Pi}(y_1, \ldots, y_d) = S(\overline{\Pi}_1(y_1), \ldots, \overline{\Pi}_d(y_d)). \quad (2)$$

If the marginal tail integrals are continuous on $[0,\infty]$, then S is unique. Otherwise, it is unique on $\mathrm{Ran}\overline{\Pi}_1 \times \cdots \times \mathrm{Ran}\overline{\Pi}_d$.
Conversely, if S is a S-copula and $\overline{\Pi}_1, \ldots, \overline{\Pi}_d$ are marginal tail integrals, then Π as defined in (2) is a joint tail-integral with marginals $\overline{\Pi}_1, \ldots, \overline{\Pi}_d$. □

Examples of S-copulas

Example [Independence S-copula]
Let $X = (X_1, \ldots, X_d)$ be a subordinator with characteristic triplet $(c, 0, \nu)$. Its componenents are independent iff

$$\nu(A) = \sum_{i=1}^{d} \nu_i(A_i) \quad A \in \mathcal{B}(\mathbb{R}^d)$$

where $A_i = \{x \in \mathbb{R} : (0, \ldots, 0, x, 0, \ldots, 0) \in A\}$, where x stands at the ith component. For the tail integral this means for $x = (x_1, \ldots, x_d)$

$$\overline{\Pi}(x) = \overline{\Pi}_1(x_1) I_{\{x_2 = \ldots = x_d = 0\}} + \cdots + \overline{\Pi}_d(x_d) I_{\{x_1 = \ldots = x_{d-1} = 0\}}.$$

This implies the S-copula

$$S_{\perp}(x_1, \ldots, x_d) = x_1 I_{\{x_2 = \ldots = x_d = \infty\}} + \cdots + x_d I_{\{x_1 = \ldots = x_{d-1} = \infty\}}. \qquad \square$$

Example [Complete dependence S-copula]
Let $X = (X_1, \ldots, X_d)$ be a subordinator with equal components. Its Lévy measure is given by $\nu(x_1, \ldots, x_d) = \nu_1(x_1) I_{\{x_1 = x_2 = \cdots = x_d\}}$. For the tail integral this means for $x = (x_1, \ldots, x_d)$

$$\overline{\Pi}(x) = \int_{\max(x_1, \ldots, x_d)} \Pi_1(u) du = \min(\overline{\Pi}_1(x_1), \ldots, \overline{\Pi}_d(x_d)).$$

This implies the S-copula

$$S_{\|}(x_1, \ldots, x_d) = \min(x_1, \ldots, x_d).$$

□

Example [Archimedian S-copula]
Let $\phi : [0,\infty]$ a strictly decreasing function with $\phi(0) = \infty$ and $\phi(\infty) = 0$. Assume furthermore that $\phi^{\leftarrow}$ has derivatives up to order d with $(-1)^k \frac{d^k \phi^{\leftarrow}(t)}{dt^k} > 0$ for $k = 1, \ldots, d$. Then the following is an S-copula

$$S(x_1, \ldots, x_d) = \phi^{\leftarrow}(\phi(x_1) + \cdots + \phi(x_d)).$$

□

Example [Clayton S-copula]
Take $\phi(t) = t^{-\theta}$ for $\theta > 0$. Then the Archimedian S-copula

$$S_\theta(x_1, \ldots, x_d) = (x_1^{-\theta} + \cdots + x_d^{-\theta})^{-1/\theta}$$

is called Clayton S-copula. Note that
$\lim_{\theta\to\infty} S_\theta(x_1, \ldots, x_d) = S_{\|}(x_1, \ldots, x_d)$
$\lim_{\theta\to 0} S_\theta(x_1, \ldots, x_d) = S_{\perp}(x_1, \ldots, x_d)$ □

Proposition [Lévy measure of bivariate sum]
Let $X = (X_1, X_2)$ be a bivariate subordinator and $X^+ = X_1 + X_2$. Assume that the Lévy measures Π_1, Π_2 are absolutely continuous on $(0, \infty)$ and that the S-copula is two times continuously differentiable on $(0, \infty)^2$ satisfying $\frac{\partial S(u,v)}{\partial u}|_{u=\overline{\Pi}_1(x), v=0} = 0$. Then X^+ has tail integral

$$
\begin{aligned}
\overline{\Pi}^+(z) &= \Pi^+([z, \infty)) \\
&= \Pi(\{(x, y) \in (0, \infty)^2 : x + y \geq z\}) + \overline{\Pi}(\{0\} \times [z, \infty)) + \overline{\Pi}([z, \infty) \times \{0\}) \\
&= \int_0^\infty \int_{(z-x)_+}^\infty \frac{\partial^2 S(u, v)}{\partial u \partial v}|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2(y)} \Pi_1(dx)\Pi_2(dy) \\
&\qquad + \overline{\Pi}(\{0\} \times [z, \infty)) + \overline{\Pi}([z, \infty) \times \{0\}) \\
&= \int_0^\infty \frac{\partial S(u, \overline{\Pi}_2((z-x)_+))}{\partial u}|_{u=\overline{\Pi}_1(x)} \Pi_1(dx) + \overline{\Pi}(\{0\} \times [z, \infty)) + \overline{\Pi}([z, \infty) \times \{0\}).
\end{aligned}
$$

Bivariate compound Poisson model

Let X_1, X_2 be compound Poisson processes both with Poisson intensity λ and claim size distribution function F. Assume that X_1, X_2 have a Clayton S-copula with parameter $\theta > 0$. Then

$$\overline{\Pi}^{+}(z) = \lambda(I_1(x) + I_2(z) + 2I_3(z)), \quad z > 0,$$

where

$$\begin{aligned}
I_1(z) &= \int_0^z \overline{F}^{\theta+1}(z-x)\left(\overline{F}^{\theta}(x) + \overline{F}^{\theta}(z-x)\right)^{-\frac{\theta+1}{\theta}} F(dx) \\
I_2(z) &= \int_z^\infty \left(\overline{F}^{\theta}(x) + 1\right)^{-\frac{\theta+1}{\theta}} F(dx) \sim \overline{F}(z) \quad z \to \infty \\
I_3(z) &= \overline{F}(z)(1 - (1 + \overline{F}^{\theta}(z))^{-1/\theta}) = o(\overline{F}(z)) \quad z \to \infty.
\end{aligned}$$

Idea: Define

$$\widetilde{\lambda} = \overline{\Pi}^{+}(0) = 2\lambda(1 - 2^{-1/\theta}) + \lambda \int_0^1 (v^\theta + \lambda^{-\theta})^{-\frac{\theta+1}{\theta}} dv.$$

Then $X^+ = X_1 + X_2$ can be identified with a **compound Poisson process** with rate $\widetilde{\lambda}$ and claim size distribution with tail

$$\overline{G}(z) = (\lambda/\widetilde{\lambda})(I_1(z) + I_2(z) + 2I_3(z)), \quad z \geq 0$$

satisfying $\overline{G}(0) = 1$.

Pareto models:

$$\overline{F}(x) = \left(\frac{a}{a+x}\right)^b, \quad x > 0, \quad a > 0, b > 1,$$

We find for any $\theta > 0$ such that $b\,\theta \geq 1$ (by lengthy calculations):

$$\overline{\Pi}^+(z) = \lambda(z^{-b}\ell(z) + 2\overline{F}(z) + o(\overline{F}(z))), \quad z \to \infty,$$

in particular, $\overline{\Pi}^+(\cdot) \in \mathcal{R}_{-b}$.

Assume that the net profit condition $c - 2\lambda EY > 0$ holds. Then

$$\Psi(u) \sim \frac{2\lambda}{c - 2\lambda EY}\,\frac{u}{b-1}\,(\overline{F}(u) + \frac{1}{2}I_1(u)), \quad u \to \infty,$$

where $\overline{F}(\cdot) + \frac{1}{2}I_1(\cdot) \in \mathcal{R}_{-b}$; in particular, $\Psi \in \mathcal{R}_{-(b-1)}$.

Independent Pareto model:

$$\overline{\Pi}^{+}(z) = \overline{\Pi}_1(z) + \overline{\Pi}_2(z) = 2\lambda \overline{F}(z), \quad z \geq 0.$$

This implies for the ruin probability as $u \to \infty$,

$$\Psi_{\perp}(u) = \frac{2\lambda}{c - 2\lambda EY} \int_u^\infty \overline{F}(z) dz \sim \frac{2\lambda}{c - 2\lambda EY} \frac{u}{b-1} \overline{F}(u).$$

Complete dependent Pareto model:

$$\overline{\Pi}^{+}(z) = \overline{\Pi}_1(z/2) = \lambda \overline{F}(z/2), \quad z \geq 0.$$

This implies for the ruin probability as $u \to \infty$,

$$\Psi_{\parallel}(u) \sim \frac{2\lambda}{c - 2\lambda EY} \int_u^\infty \overline{F}(z) dz \sim \frac{2\lambda}{c - 2\lambda EY} \frac{u}{b-1} 2^{b-1} \overline{F}(u) \succeq \Psi_{\perp}(u).$$

Clayton-Pareto model, $b\theta = 1$:

$$\overline{\Pi}^{+}(z) \sim 2\lambda(b+1)\overline{F}(z), \quad z \to \infty.$$

This implies for the ruin probability as $u \to \infty$,

$$\Psi(u) \sim \frac{2\lambda}{c - 2\lambda EY}\frac{u}{b-1}(b+1)\overline{F}(u) \; > \; 2\Psi_{\perp}(u).$$

Comparisn with the complete dependent Pareto model depends on the value of b.

Exponential models:

$$\overline{F}(x) = \exp(-ax), \quad x > 0, \quad a > 0.$$

Consider $\theta = 1$. Set $t = e^{az}$:

$$\begin{aligned} I_1(z) &= \int_{e^{-az}}^{1} \frac{dy}{(1+ty^{2\theta})^{\frac{\theta+1}{\theta}}} = \frac{1}{2} e^{-\frac{1}{2}az} \left(\arctan e^{\frac{1}{2}az} - \arctan e^{-\frac{1}{2}az} \right) \\ &\leq e^{-\frac{1}{2}az}, \quad z \geq 0, \end{aligned}$$

and $I_1(z) \sim \frac{\pi}{4} e^{-\frac{1}{2}az}, \quad z \to \infty.$
Moreover,

$$I_2(0) = \int_0^1 (1+v)^{-2} dv = \frac{1}{2} \quad \text{and} \quad I_2(z) \leq e^{-az}, \quad z \geq 0.$$

Finally, $I_3(0) = 1/2$.

Then $X^+ = X_1 + X_2$ can be identified with a **compound Poisson process** with

$$\widetilde{\lambda} = \overline{\Pi}^+(0) = \lambda(I_1(0) + I_2(0) + 2I_3(0)) = (3/2)\lambda$$

and claim size distribution G with tail

$$\overline{G}(z) \leq e^{-\frac{1}{2}az} + 2e^{-az}, \quad z > 0.$$

Assume that the net profit condition $c - 2\lambda EY > 0$ holds.
If the independent model has a Lundberg coefficient: $\exists \kappa_\perp$ s.t.

$$\widehat{f}_I(\kappa_\perp) = \int_0^\infty e^{\kappa_\perp x}\overline{F}(x)dx = \frac{c}{2\lambda},$$

then for $K_1, \kappa_1 > 0$

$$\Psi(u) \sim K_1 e^{-\kappa_1 u}, \quad u \to \infty,$$

If $\kappa_{||}$ is the Lundberg coefficient of the completely dependent model, then $\kappa_{||} = \frac{1}{2}\kappa_\perp$. If $c - 4\lambda EY > 0$, then

$$0 < \kappa_{||}\,, \kappa_1 < \kappa_\perp < a.$$