Lifetime Consumption and Investment: Retirement and Constrained Borrowing*

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Abstract

Endogenous choice of when to retire has an interesting impact on optimal portfolio choice and consumption, whether or not it is possible to borrow against labor income. When retirement is voluntary, human capital is negatively correlated with the stock market even when the wage itself is not. This negative correlation implies more stock investment than with a mandatory exogenous retirement date. Portfolio choice can jump down at voluntary retirement; consumption can jump up or down. Inability to borrow limits hedging and reduces the value of labor income, the wealth-to-wage ratio threshold for retirement, and the stock investment. If the wage correlates positively enough with the market, stock investment may start negative and increase over time, even when the risk premium is positive. This contradicts brokers' traditional advice that young investors should be aggressive and older investors should be conservative.

Journal of Economic Literature Classification Numbers: D11, D91, G11, C61.

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I. Introduction

Retirement is one of the most important economic events in a worker's life. Not surprisingly, retirement is connected to a number of important personal decisions such as consumption and investment and also to policy issues such as those on insurance and pensions, as well as mandatory versus voluntary retirement.[1] In this paper, we extend recent advances in finance to build a tractable optimal consumption and investment model with voluntary or mandatory retirement, and with or without a non-negative wealth constraint (which prevents borrowing against future wages). This paper solves three models for which more or less complete solutions are available. A companion piece studies explicit dependence of the mortality rate, wage, and preference for working on the stage of life. It is hoped that these models and extensions will be useful for studying policy questions in insurance and retirement.

We consider three models in our analysis. The three models vary in the treatment of retirement and borrowing against future labor income. All three models share a number of common features: a constant hazard rate of mortality, different preferences for consumption before and after retirement, possibly stochastic labor income, bequest, and actuarially fair life insurance. Keeping these features the same makes it easy to perform a parallel comparison of the three models. All three models consider retirement to be irreversible, emphasizing that a worker may be much more valuable to a firm working full-time than working part-time, as pointed out by Gustman and Steinmeier (1986). This is an extreme alternative to models with a continuously variable labor-leisure choice, as in Liu and Neis [2002]. We have also looked at models with both types of choice, allowing the possibility of a return to part-time work at a lower wage after retirement, but not in this paper.

The first model is a benchmark case with a fixed retirement date, which we interpret as mandatory retirement.[2] Our first model is a close relative of the Merton [1969] model with i.i.d. returns and constant relative risk aversion, and it can be solved explicitly.

The second model has voluntary retirement in a model in which the agent is free to borrow against future labor income. The second model is solved explicitly in the dual (as a function of the dual variable which is the marginal utility of wealth in the value function). This is an explicit parametric solution of the original problem which means that we know everything about the solution once we have conducted a one-dimensional numerical search for the value of the dual variable that corresponds with the current wealth level. Before retirement, there is a critical wealth-to-wage ratio at which it is optimal to retire. The expected time to retirement depends on what the wage and what the financial wealth are today: the agent self-insures against risk in the security market by working more when the security market returns are poor. By working longer in expensive states, the agent generates more income (some of which is transferred to other states) for the average time worked than if the agent worked for the same amount of time in each state. Because of the self-insurance

[1] In the UK, mandatory retirement is still widespread, and this is still an active policy issue (see Meadows [2003]). In the US, the Age Discrimination in Employment Act of 1967 (ADEA) generally prohibits mandatory retirement. One exception is for a qualifying "bona fide executive" or person in a "high policymaking position" who can face mandatory retirement at an age of 65 or above. There used to be an exception for tenured academics who could face mandatory retirement at an age of 70 or above, but that exception expired on January 1, 1994.

[2] A more realistic model of mandatory retirement is given by Panageas and Farhi [2003], who permit retirement at or before mandatory retirement date. Our simpler assumption is better for our benchmarking because we can solve the model exactly and it is easier to compare with the other models.

role of voluntary retirement, the agent invest a greater fraction of the total wealth (i.e., financial wealth plus human capital) in the stock market compared to the benchmark model.

The third model has voluntary retirement in a model in which the agent cannot borrow against future labor income. This restriction reduces the usefulness of investing in stocks because any significant negative return would wipe out the financial wealth and bring the agent against the borrowing constraint. Accordingly, the agent invests a smaller fraction of the total wealth in the stock market compared to the second model. In addition, the borrowing constraint prevents the agent from transferring income across states to the extent that would be optimal and reduces the attractiveness of working longer in expensive states of nature. As a result, the critical wealth-to-wage ratio at which it is optimal to retire is lower in the third model than that in the second model.

The paper contains technical innovations that permit solution up to determination of a few parameters. In particular, we combine the dual approach of Pliska (1986), He and Pagès (1993), and Karatzas and Wang (2000) with an analysis of the boundary to obtain a problem we can solve in parametric form even if no known solution exists in the primal problem.

Consumption and portfolio choice can both jump at the endogenous retirement date in the second and the third model. Consumption can jump because preferences are different after retirement. This could be due to household production (time to cook instead of buying more expensive prepared food), reduced work-related expenses (for clothes or commuting), or just different preference for consumption when more leisure is available. Portfolio choice jumps at retirement because there is significant hedging of human capital just before retirement. When just below the critical wealth for retirement, the agent knows that increase in wealth implies human capital goes to zero (or a small post-retirement level) but a decrease in wealth implies human capital may be much greater because of a possibly long excursion through the no-retirement region. Hence, human capital before retirement has substantial negative correlation with the market return. Therefore, the portfolio holding in the market is lower just after retirement than just before retirement because of the disappearance of the human capital hedge at retirement.

In general, human capital is a less and less important fraction of total wealth over time, and the fraction of financial wealth in risky assets will vary to counteract the risk exposure in human capital. If the wage is uncorrelated with the market, human capital is underexposed to market risk (and indeed will have negative exposure to the market once we include the impact of varying the duration of employment as the market goes up and down), and the risky asset position will be larger when young than when old. On the other hand, if the wage correlates positively enough with the market, then the market exposure in human capital becomes positive and is greater for the young than for the old, even with voluntary retirement. Consequently, the optimal risky asset position will be smaller when young than when old. In fact, the optimal risky asset position can even become negative for the young, even when the risk premium is positive, if the market exposure in the human capital is more than what is optimal. This implication of our model contradicts brokers' traditional advice that young investors should be aggressive and older investors should be conservative.

Lachance (2004) also considers the optimal consumption and investment problem with endogenous retirement. She also finds that endogenous retirement can serve as a self-insurance against a market downturn and can thus increase stock investment before retirement. However, she allows the investor to borrow against all future labor income and does not consider any life insurance or bequest. As we show in this paper, ignoring borrowing constraint would overstate the hedging benefit of human capital and thus the optimal stock investment before retirement. Liu and Neis (2002) consider the optimal consumption and investment problem with endogenous working hours. In contrast to our model, they allow an investor to borrow against future labor income. In addition, they assume that the stock price can never fall below a fixed positive level. Bodie, Merton, and Samuelson (1992) consider the effect of labor choice on optimal investment policy and Basak (1999) develops a continuous-time general equilibrium model to adapt dynamic asset pricing theory to include labor income. Dowell and McLaren (1986) construct a deterministic labor choice model allowing borrowing fully against future labor income. Similar to Liu and Neis (2002), these papers assume that working hours are infinitely divisible. Sundaresan and Zapatero (1997) examine how pension plans affect the retirement policies with an emphasis on the valuation of pension obligations. They abstract from modelling the disutility of working and the investor's investment opportunities outside the pension. Khitatrakun (2002) shows that individuals not constrained by institutional constraints respond to a positive wealth shock by retiring or expecting to retire early than previously expected. Gustman and Steinmeier (2001) find that there is a positive correlation between wealth and retirement. Stock and Wise (1990) examine the effects of firm pension plan provisions on the retirement decisions and the option value of work. Mitchell and Fields (1984) find that retirement age differences are due in part to differences in worker preferences and in part to differences in income opportunities conditional on pension rules.

The rest of the paper is organized as follows. Section II. presents the formal choice problems used in most of the paper. Section III. presents graphically the solution presented in Section IV. as well as a numerical solution of the case with locally unspanned labor income described in Section V.. Section VI. closes the paper. All of the proofs are in the appendix.

II. Choice Problems

Our general goal is to provide a tractable workhorse model that can be used to analyze various issues related to life cycle consumption and investment, retirement, and insurance. This paper focuses on stationary models that can be solved more or less explicitly. A companion piece looks at models with life stages having potentially different hazard rates of mortality, disutilities of working, incidence of sickness, and pure rate of time discount. This section poses the formal decision problems for most of the stationary models considered in this paper.

The choice problems make many of the assumptions that are common in continuous-time financial models, for example the constant riskfree rate and lognormal stock returns. Other assumptions are not standard but seem particularly appropriate for analysis of life-cycle consumption and investment. For example, our model includes mortality and bequest as well as preference for not working. In addition, before retirement an agent earns labor income with potentially stochastic wage and can also purchase life insurance or term annuity.

All the models in this paper consider pure retirement without flexible hours, return to full-time work in retirement, or part-time work in retirement. There is no reason why these other features cannot be added to the model, but we choose to focus instead on the essential nonconvexity that says half-time work is much less valuable than full-time work in some positions, as also pointed out by Gustman and Steinmeier (1986).

In our main analysis we consider the following three cases:

benchmark fixed retirement date and free borrowing against wages (Problem 1 and Theorem 1).

NBC ("No Borrowing Constraint") free choice of retirement date and free borrowing against wages (Problem 2 and Theorem 2).

BC ("Borrowing Constraint") free choice of retirement date but no borrowing against wages (Problem 3 and Theorem 3).

The benchmark case is a close relative of the Merton [1969] model with i.i.d. returns and constant relative risk aversion. Moving to the NBC case isolates the impact of making retirement flexible. Subsequently moving to the BC case isolates the impact of the borrowing constraint.

Here are the three choice problems corresponding to the above three cases.

Problem 1 (benchmark) *Given initial wealth W_0, initial income from working y_0, and time-to-retirement T with associated retirement indicator $R_t = \iota(T \leq t)$ for some fixed time-to-retirement T, choose adapted nonnegative consumption $\{c_t\}$, adapted portfolio $\{\theta_t\}$, and adapted nonnegative bequest $\{B_t\}$, to maximize expected utility of lifetime consumption and bequest*

$$E\left[\int_{t=0}^{\infty} e^{-(\rho+\delta)t}\left((1-R_t)\frac{c_t^{1-\gamma}}{1-\gamma} + R_t\frac{(Kc_t)^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_t)^{1-\gamma}}{1-\gamma}\right)dt\right] \tag{1}$$

subject to the budget constraint

$$\begin{aligned} W_t \;=\; & W_0 + \int_{s=0}^{t}(rW_s ds + \theta_s^\top((\mu - r\mathbf{1})ds + \sigma^\top dZ_s) + \delta(W_s - B_s)ds - c_s ds \\ & + (1-R_s)y_s ds), \end{aligned} \tag{2}$$

the labor income process

$$y_t \equiv y_0 \exp\left((\mu_y - \frac{{\sigma_y}^\top \sigma_y}{2})t + {\sigma_y}^\top Z_t\right), \tag{3}$$

and no-borrowing-without-repayment

$$W_t \geq -g(t)y_t, \tag{4}$$

where

$$g(t) \equiv \begin{cases} \left(\frac{1-e^{-\beta_1(T-t)}}{\beta_1}\right)^+ & \beta_1 \neq 0 \\ (T-t)^+ & \beta_1 = 0, \end{cases} \tag{5}$$

$$\beta_1 \equiv r + \delta - \mu_y + {\sigma_y}^\top \kappa \tag{6}$$

is the effective discount rate for labor income, and

$$\kappa \equiv (\sigma^\top)^{-1}(\mu - r\mathbf{1}) \tag{7}$$

is the price of risk.

Problem 2 (NBC) *Given initial wealth W_0, initial income from working y_0, and initial retirement status R_{0-}, choose adapted nonnegative consumption $\{c_t\}$, adapted portfolio $\{\theta_t\}$, adapted nonnegative bequest $\{B_t\}$, and adapted nondecreasing retirement indicator*[3] *$\{R_t\}$, to maximize*

[3] By "indicator," we mean a right-continuous process taking values 0 and 1, and by nondecreasing we mean that for $0 < s < t$, $R_{0-} \leq R_0 \leq R_s \leq R_t$.

expected utility of lifetime consumption and bequest (1) subject to the budget constraint (2), labor income before retirement (3), and no-borrowing-without-repayment

$$W_t \geq -(1 - R_t)\frac{y_t}{\beta_1}, \tag{8}$$

where β_1 is assumed to be positive.

Problem 3 (BC) *The same as Problem 2, except that the no-borrowing-without-repayment constraint is replaced by the stronger non-negative wealth constraint*

$$W_t \geq 0. \tag{9}$$

The uncertainty in the model comes from two sources: the standard Wiener process Z_t and the Poisson arrival of mortality at a fixed hazard rate δ. These are drawn independently. The Wiener process Z_t has dimensionality equal to the number of linearly independent risky returns, and maps into security returns through the constant mean vector μ and the constant nonsingular standard deviation matrix σ giving the sensitivities of the various securities to the underlying uncertainty Z. For most of the paper, we will assume that local changes in the labor income y are spanned by local returns on the assets, but we will generalize this result (requiring a numerical solution) in Section V and provide some plots in Section III. The common objective function (1) for the problems has already integrated out the impact of mortality risk: utility is discounted at the rate $\rho + \delta$ where ρ is the pure rate of time discount and δ is the hazard rate of mortality. Some early work on utility theory suggested that the pure rate of time discount is positive only because of the effect of mortality; if this is true then we could take ρ to be 0. Still, the problem would not be the same as the traditional problem due to the presence of bequest and insurance. Insurance is assumed to be fairly priced at the rate δ per unit of coverage, both long and short. When $W - B > 0$, this is term life coverage purchased for a premium of $\delta(W - B)$ per unit time. If $W - B < 0$, then this is a short position in term life coverage, which is like a term version of a life annuity since it trades wealth in the event of death for more consumption when living.

Retirement status affects both preferences and income. The retirement status at time t is given by the retirement indicator R_t, which is 1 after retirement and 0 before retirement. Retirement is exogenously fixed at time T in Problem 1 and endogenously chosen in Problems 2 and 3. In all cases, retirement is right-continuous (technically convenient) and nondecreasing (retirement is irreversible). The state variable R_{0-} is the retirement status at the beginning of the investment horizon. This may be different from R_0 since if not retired at the outset ($R_{0-} = 0$), it may still be optimal to retire immediately ($R_0 = 1$). We take retirement to be irreversible: for $0 < s < t$, $R_{0-} \leq R_0 \leq R_s \leq R_t$. Irreversible retirement is not the only modeling choice. For example, completely flexible hours have been considered by Liu and Neis [2002]. In their model, the agent can move freely in and out of retirement, and can vary hours continuously when working. We focus instead on the non-convexity of working: there are lot of jobs in which working half-time is worth a lot less to the company than working full-time. We consider an extreme case in which the worker either works full-time or does not work at all. It is not hard to solve intermediate cases, for example, we have solved an example in which the worker can only work full-time before retirement (say as a sales manager) but is free to work flexible hours at a lower wage (say as receptionist in a community center) after retirement. We could also model a case in which skills depreciate slowly after retirement; perhaps it is impossible to return to the profession after five years but it is possible to return at a small decrease in wage after three months. We leave all these possibilities for elsewhere; this paper concerns only the pure case in which retirement is completely irreversible.

The utility function (1) is a standard time-separable von Neumann-Morgenstern utility function with mortality and bequest. The utility function features the same constant level $\gamma > 0$ of risk aversion for consumption before retirement, after retirement, and for bequests. Felicity of consumption or bequest is discounted using a pure rate of time discount ρ plus the mortality rate δ. The constant $K > 1$ indicates how much more consumption before retirement must be increased to compensate for having to work. Preference for not working could be due to a disutility of work, or it could be due indirectly to household production and cost savings. For example, when working there may not be enough time to shop for bargains or prepare meals or take a cruise. The constant $k > 0$ measures the intensity of preference for leaving a large bequest, the limit $k^{1-\gamma} \to 0$ implements no preference for bequest.

The terms in the integrand of the wealth equation (2) are mostly familiar. The first term says that if all wealth is invested in the riskfree asset, the rate of return is r. For the dollar investment θ_t in the risky asset, there is risk exposure ${\theta_t}^\top \sigma^\top dZ_t$ and the mean return ${\theta_t}^\top \mu dt$ is substituted for the corresponding riskfree return $r{\theta_t}^\top \mathbf{1} dt$ (where $\mathbf{1}$ is a vector of 1's with dimension equal to the number of risky assets). The term $\delta(B_t - W_t)dt$ is the insurance premium we have already discussed, $c_t dt$ is payment for consumption, and $(1 - R_t)y_t dt$ is labor income. The factor $(1 - R_t)$ multiplying wage income says income disappears after retirement.

In general, it is a subtle question what kind of constraint to include in an infinite-horizon portfolio problem to rule out borrowing without repayment and doubling strategies. Fortunately, there is a simple and reasonable constraint that suffices in our problem. The no-borrowing-without-repayment constraints (4) or (8) specify that the level of indebtedness ($= \max(-W_t, 0)$) can never be larger than earnings potential. The two constraints differ because the earnings potential is different when there is a fixed retirement date than when retirement is a choice and it is possible to work until death. In Problem 3, the no-borrowing-without-repayment constraint is replaced by the stronger no-borrowing constraint (9).

To review the differences in the problems, moving from the benchmark Problem 1 to the NBC Problem 2, the fixed retirement date T ($R_t = \iota(t \geq T)$) is replaced by free choice of when to retire (R_t a choice variable), along with a technical change in the calculation of the maximum value of future labor income ((4) to (8)). Moving from the NBC Problem 2 to the BC Problem 3 replaces the no-borrowing-without-repayment constraint (8) with a no-borrowing constraint (9).

III. Graphical Solution

It is useful to anticipate the economic content of paper by graphing the analytical solutions given in Theorems 1, 2, and 3 in Section IV. below. We present many of the results normalized by total wealth, equal to financial wealth plus human capital, where human capital is the market value of future labor income in the optimal solution. The formulas for human capital are given in Proposition 1 at the end of Section IV. below. While the market's valuation of the individual's human capital may be different from the individual's own valuation (due to the borrowing constraint), this is still a useful normalization for interpreting the results. In addition, we assume constant wage rate for the first six figures.

Recall that the results consider three cases:

benchmark fixed retirement date and free borrowing against wages (Problem 1 and Theorem 1)

NBC ("No Borrowing Constraint") free choice of retirement date and free borrowing against wages (Problem 2 and Theorem 2)

BC ("Borrowing Constraint") free choice of retirement date but no borrowing against wages (Problem 3 and Theorem 3)

Figure 1 shows the optimal stock position in the three cases, per unit of total wealth, as a function of financial wealth. The horizontal line shows the optimal portfolio choice for the benchmark case. In this case, it is as if all future wage income is capitalized and then there is fixed-proportions investment as in the Merton model. The portfolio proportion is the same constant whatever the time to retirement T is (and even after retirement). Moving to the NBC case, permitting flexibility in the retirement date permits a larger equity position because working longer can insure against variation in the stock market. The agent works longer in expensive states (when the market is down) and the wealth from these states is transferred to other states by taking a significant position in equities. With the borrowing constraint in the BC case, transfer of wealth across states is restricted and indeed there is not much point of taking a significant position in equities when financial wealth is low, since that would just imply bumping into the borrowing constraint much of the time.

Figure 2 shows the consumption rate, normalized by total wealth, as a function of financial wealth. An interesting point not visible in the picture is that consumption jumps at retirement (see Table 1), in a direction that depends on whether risk aversion is larger or smaller than 1 (log utility). In the benchmark case, the consumption rate does not depend on financial wealth, but it does depend on time to maturity (not shown). Moving to the NBC case, adding flexible retirement leads to a higher consumption rate when wealth is higher (due to risk aversion greater than 1 in the example), since high wealth implies retirement is expected soon and demand for consumption is less after retirement. Moving to the BC case, consumption is significantly less at low wealth levels, which represents precautionary savings against market declines.

The retirement threshold for the wage-normalized wealth level (i.e., wealth-to-wage ratio) at which the agent chooses to retire as a function of relative risk aversion is plotted in Figure 3. There is no curve for the benchmark case, since in that case retirement is at a fixed date, not at a freely chosen wealth boundary. The retirement threshold is lower when there is a borrowing constraint than when there is none: while retirement is equally desirable in both cases, continuing to work is less desirable when borrowing is limited. The retirement threshold decreases in risk aversion because it is valuable to earn more money to take advantage of market returns when risk aversion is small.

The value of human capital can vary inversely with wealth as a form of insurance. Figure 4 shows the dependence of human capital on financial wealth. This is a constant over wealth in the benchmark case, but would vary from zero to the maximum on the NBC curve as maturity increases. In the BC and NBC cases, we see the insurance effect of how the agent hedges financial risk by working longer (and thereby increasing the value of human capital) when financial wealth is low.

Having flexible retirement and borrowing (the NBC case) is the least constrained of the three cases. It is interesting to measure the loss in value in the other two cases. Figure 5 gives the value loss from losing retirement flexibility, as a function of the fixed time to retirement. The value loss is measured as a certainty-equivalent fraction of total wealth corresponding to moving from the NBC case to the benchmark case. When wealth is high (=20), retiring soon is optimal and the loss is least when forced retirement comes soon. As wealth decreases, it becomes optimal to work longer and longer and the minimum loss is at larger and larger fixed times to retirement. Figure 6 shows the value of being able to borrow measured in the certainty-equivalent fraction of total wealth as a function of the financial wealth. The value is small when financial wealth is large and is large when the borrowing constraint is nearly binding. The value of being able to borrow is greater when the mean return on the stock is higher, since being able to borrow makes it possible to make full use of equity.

Stock brokers have traditionally advised customers that young people should take on more risk than older people who are close to retirement. Our analysis can be used to generalize and confirm analysis of Jagannathan and Kocherlakota [1996] that calls into question this traditional rule. We have already seen in Figure 1 that in both the NBC case and BC case, the risky asset holding increases as a function of wealth, and higher wealth corresponds to being nearer to retirement. Arguably this result is not a fair criticism of the traditional advice because of the normalization by total wealth. Figure 7 shows the portfolio choice normalized by financial wealth (for the NBC case), which may be closer to the intent of the traditional advice. When wages are less risky (top curves), the proportion of financial wealth put in stock does indeed fall as financial wealth increases, but this result is reversed in the case of an agent whose salary has a large positive sensitivity to market risk (bottom curves). The dotted curves in Figure 7 show a case (described in Section V. in which there is idiosyncratic volatility in income so that the labor income is not locally spanned by market returns. Not having spanning tends to moderate hedging demand for the risky asset in the portfolio.

Part of the problem with the traditional advice can be understood by the benchmark case in which risky asset position is chosen to be a fixed proportion of total wealth. When young, most of the agent's total wealth is in human capital, and if the wage is positively correlated with market this may already represent too much exposure to market risk, implying a desire for to take a short position in stocks to hedge the excess risk. When retirement is discretionary, the stock choice is less for the direct reason we have been discussing that wages are riskier, and also because it is less attractive to work more in expensive states (since those are also states of low wage when wages move in the same direction as stocks). Together with the result that a borrowing constraint reduces stock demand at low levels even more, these results question the usefulness of the traditional advice, at least in the absence of a lot of qualifications of when the advice should be applied.

Table 1 contains a number of solutions illustrating the sensitivity of the equilibrium to various parameters. In particular, it shows the significant drop of the consumption and the stock investment at retirement date. In addition, as the mortality rate decreases, the investor requires a higher retirement threshold (i.e., higher wealth-to-wage ratio) to retire and consumes less to save for after-retirement.

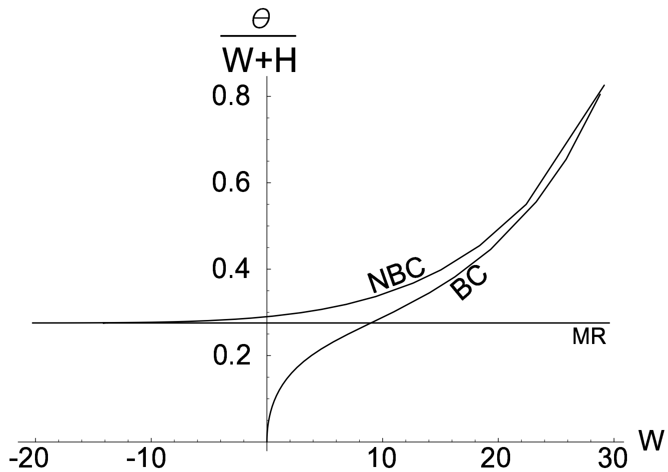

Figure 1: Equity per total wealth given financial wealth

Total wealth equals financial wealth W plus human capital H. The horizontal line is for the benchmark case with a fixed retirement date 20 years from now and free borrowing against future wages. The NBC ("No Borrowing Constraint") case adds free choice of retirement date but still allows free borrowing. The BC ("Borrowing Constraint") case adds a nonnegative wealth constraint that restricts borrowing against future wages. The plot is based on parameters $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

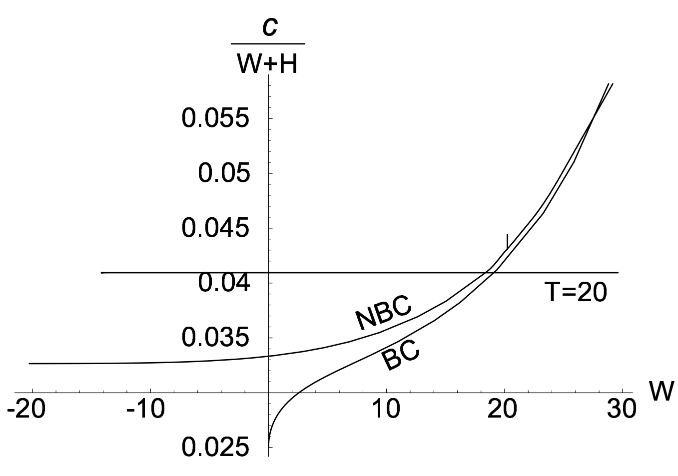

Figure 2: Consumption per total wealth given financial wealth

Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

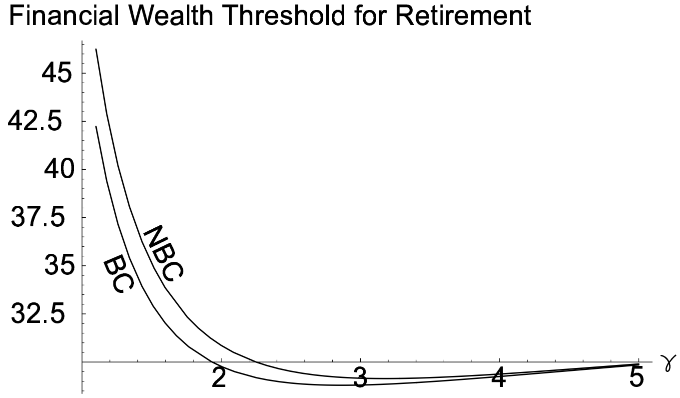

Figure 3: Retirement threshold given risk aversion

The agent retires the first time wage-normalized financial wealth reaches the retirement threshold. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

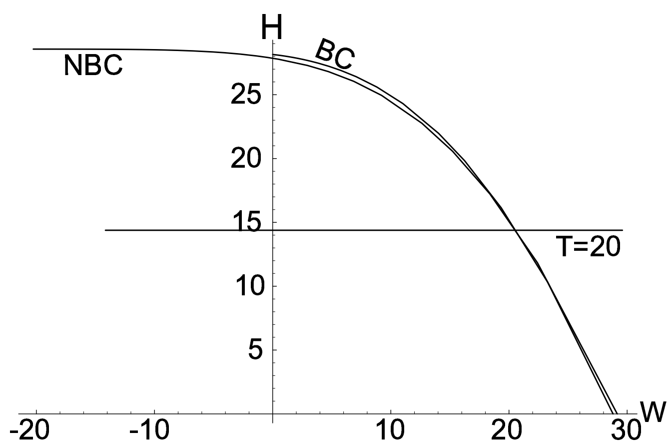

Figure 4: Human capital given financial wealth

The smaller the financial wealth, the longer it is expected to work and the higher the human capital. For this reason, human capital has a negative beta. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

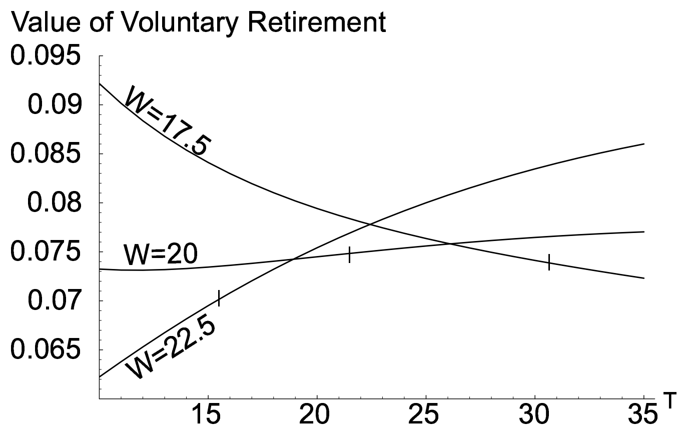

Figure 5: Value of voluntary retirement given horizon

The value is measured as the fraction by which wealth would have to be increased to compensate for a change from free choice of retirement to retirement at the fixed date T. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

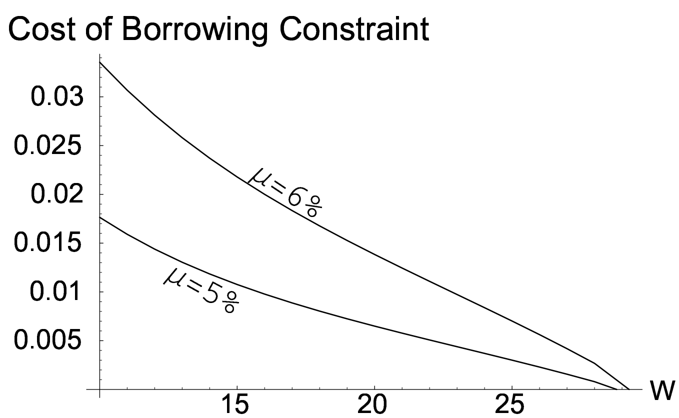

Figure 6: Value of borrowing given financial wealth

The value is measured as the fraction by which wealth would have to be increased to compensate for not having access to borrowing. Parameters: $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

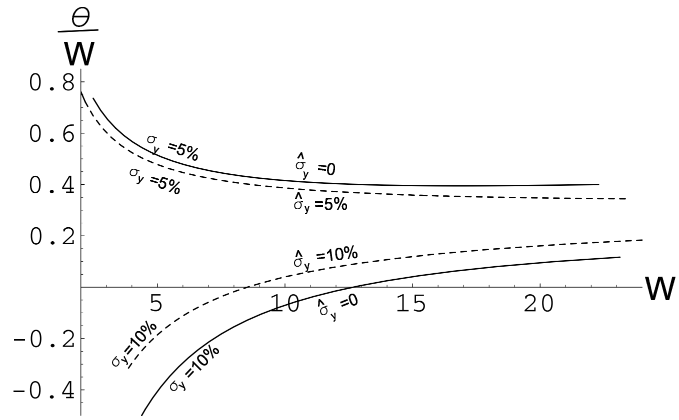

Figure 7: Equity holdings and risky labor income

When labor income moves with the market, risky asset demand is reduced for the direct reason that human capital is part of the agent's portfolio and for an indirect reason that the agent is less inclined to work longer when the market is low. Uncorrelated risk has a small impact on the equity holdings. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $\mu_y = 0$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, and $y_0 = 1$.

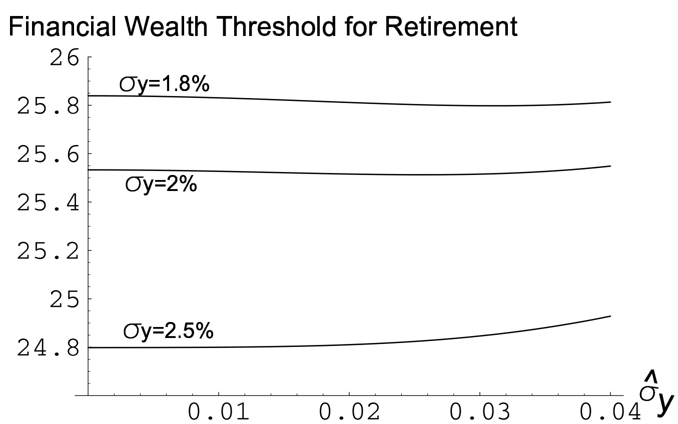

Figure 8: Retirement threshold and risky labor income

The retirement threshold is relatively insensitive to market risk (different curves) and idiosyncratic risk (horizontal axis) in labor income. Parameters: $\mu = 0.05$, $\sigma = 0.22$, $\mu_y = 0$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, and $y_0 = 1$.

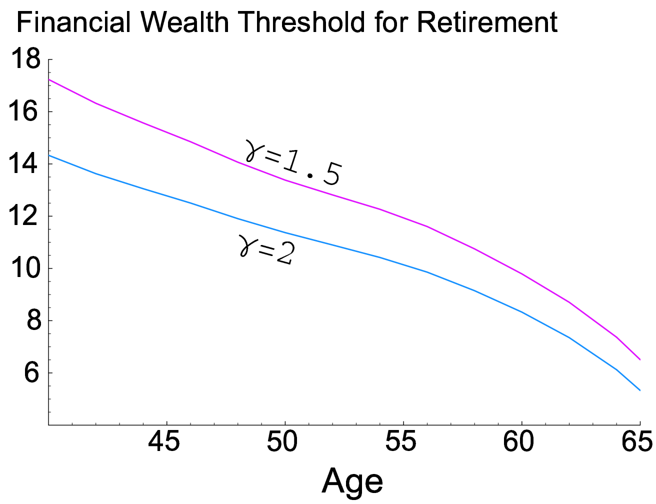

Figure 9: The retirement threshold against age.
This is based on a numerical solution of a model not presented in the paper in which the hazard rate of retirement, average wage, and preference for working all depend on age.

Parameter	$\bar{W}$	$\frac{c(\tau)}{\bar{W}}$	$\frac{c(\tau^+)}{\bar{W}}$	$\frac{\theta(\tau)}{\bar{W}}$	$\frac{\theta(\tau^+)}{\bar{W}}$	$\bar{W}_{NBC}$	$\frac{\theta_{NBC}(\tau^+_{NBC})}{\bar{W}_{NBC}}$
Base Case	28.81	0.058	0.028	0.8	0.28	29.16	0.83
$\gamma = 3.5$	28.98	0.057	0.026	0.69	0.24	29.19	0.71
$\gamma = 3.5$	28.91	0.058	0.03	0.95	0.33	29.52	0.98
$\mu = 0.04$	28.03	0.056	0.027	0.72	0.21	28.14	0.73
$\mu = 0.06$	29.26	0.061	0.029	0.89	0.34	30.06	0.92
$\sigma = 0.15$	29.38	0.065	0.031	1.40	0.59	30.76	1.48
$\sigma = 0.30$	27.97	0.060	0.03	0.53	0.15	28.07	0.53
$\delta = 0.02$	28.81	0.058	0.028	0.80	0.28	29.16	0.83
$\delta = 0.03$	20.96	0.075	0.036	0.81	0.28	21.01	0.82
$\rho = 0.008$	29.08	0.057	0.027	0.78	0.28	29.39	0.80
$\rho = 0.012$	28.54	0.059	0.028	0.83	0.28	28.94	0.85
$K = 2.5$	35.93	0.053	0.029	0.71	0.28	36.22	0.72
$K = 3.5$	24.45	0.063	0.027	0.89	0.28	24.85	0.92
$k = 0.025$	33.59	0.050	0.024	0.74	0.28	33.91	0.75
$k = 0.075$	26.87	0.062	0.030	0.84	0.28	17.86	0.76
$\mu_y = 0.01$	34.82	0.058	0.028	1.10	0.28	38.02	1.23
$\mu_y = -0.01$	25.56	0.058	0.028	0.58	0.28	25.57	0.58

Table 1: Comparative Statics

Base case parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

IV. The Analytical Solution

Let

$$\nu \equiv \frac{\gamma}{\rho + \delta - (1-\gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma})}. \tag{10}$$

For our solutions, we will assume $\nu > 0$, which is the condition for the corresponding Merton problem with positive initial wealth up front to have a solution. If relative risk aversion $\gamma > 1$, ν is always positive, but for $1 > \gamma > 0$, whether ν is positive depends on the other parameters.

Theorem 1 (benchmark) *Suppose $\nu > 0$ and that the no-borrowing constraint is satisfied with strict inequality at the initial values:*

$$W_0 > -g(0)y_0. \tag{11}$$

Then in the solution to the investor's Problem 1, the optimal wealth process is

$$W_t^* = f(t) y_t x_t^{-1/\gamma} - g(t) y_t, \tag{12}$$

the optimal consumption policy is

$$c_t^* = K^{-bR_t} f(t)^{-1}(W_t^* + g(t)y_t), \tag{13}$$

the optimal trading strategy is

$$\theta_t^* = \frac{(\sigma^\top \sigma)^{-1}(\mu - r\mathbf{1})}{\gamma}(W_t^* + g(t)y_t) - \sigma^{-1}\sigma_y g(t) y_t, \tag{14}$$

and the optimal bequest policy is

$$B_t^* = k^{-b} f(t)^{-1}(W_t^* + g(t)y_t), \tag{15}$$

where

$$b \equiv 1 - 1/\gamma, \tag{16}$$

$$f(t) \equiv (\hat{\eta} - \eta)\exp(-\frac{1 + \delta k^{-b}}{\eta}(T-t)^+) + \eta, \tag{17}$$

$$\eta \equiv (1 + \delta k^{-b})\nu, \tag{18}$$

$$\hat{\eta} \equiv (K^{-b} + \delta k^{-b})\nu. \tag{19}$$

$$x_t \equiv \left(\frac{W_0 + g(0)y_0}{y_0 f(0)}\right)^{-\gamma} e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t + \sigma_x^\top Z_t}, \tag{20}$$

$$\mu_x \equiv -(r-\rho) - \frac{1}{2}\gamma(1-\gamma)\sigma_y^\top \sigma_y + \gamma\mu_y - \gamma{\sigma_y}^\top \kappa, \tag{21}$$

and

$$\sigma_x \equiv \gamma\sigma_y - \kappa. \tag{22}$$

Furthermore, the value function for the problem is

$$V(W,y,t) = f(t)^\gamma \frac{(W+g(t)y)^{1-\gamma}}{1-\gamma}.$$

PROOF. The approach of the proof is to use a separating hyperplane to separate preferred consumptions from the feasible consumptions. Feasibility of the claimed optimum follows from direct substitution. Given (11), W^* is well-defined by (12). It is tedious but straightforward to verify the budget equation (2) using Itô's lemma and the claimed form of the strategy $(c^*, \theta^*, B^*, W^*)$ in (12)–(15), various definitions (5)–(7) and (12)–(22), and the definition of labor income (3). Note that $W_0^* = W_0$ by (12) and (20). The no-borrowing-without-repayment constraint (4) follows from positivity of $f(t)$ and x, and the definition of W^* in (12).

We start by noting the state-price density and pricing results, both for labor income and for consumption and bequest. Define the state price density process ξ by

$$\xi_t \equiv e^{-(r+\delta+\frac{1}{2}\kappa^\top\kappa)t - \kappa^\top Z_t}. \tag{23}$$

This is the usual state-price density but adjusted to condition on living, given the mortality rate δ and fair pricing of long and short positions in term life insurance.

As shown in Lemma 1 in the Appendix, $g(t)y_t$ is the value at t of subsequent labor income, where g is defined in (5). (Note that $g(t) \equiv 0$ for $t \geq T$.) Furthermore, by Lemma 1, we have that the present value of future consumption and bequest is no larger than initial wealth:

$$E[\int_0^\infty \xi_t(c_t + \delta B_t)dt] \leq W_0 + g(0)y_0, \tag{24}$$

for any feasible strategy, with equality for our claimed optimum.

We then have, for any feasible strategy (c, θ, B),

$$\begin{aligned}
& E[\int_0^\infty e^{-(\rho+\delta)t}\left(\frac{(K^{R_t}c_t)^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_t)^{1-\gamma}}{1-\gamma}\right)dt] \\
&\leq\ E[\int_0^\infty e^{-(\rho+\delta)t}\left(\frac{(K^{R_t}c_t^*)^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_t^*)^{1-\gamma}}{1-\gamma}\right)dt] \\
&\quad + \frac{x_0}{y_0^\gamma}E[\int_0^\infty \xi_t(c_t + \delta B_t - c_t^* - \delta B_t^*)dt] \\
&\leq\ E[\int_0^\infty e^{-(\rho+\delta)t}\left(\frac{(K^{R_t}c_t^*)^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_t^*)^{1-\gamma}}{1-\gamma}\right)dt],
\end{aligned} \tag{25}$$

where the first inequality follows from Lemma 2 and the second inequality follows from pricing (24) for all strategies and equality for the claimed optimum. This says that the claimed optimum dominates all other feasible strategies. We showed previously that the claimed optimum is feasible, so it must indeed be optimal.♣

Unlike Problem 1, Problems 2 and 3 do not seem to have explicit solutions in terms of the primal variables. However, we provide explicit solutions (up to at most one constant) in terms of marginal utility in Theorems 2 and 3. Recall the definition of ν in (10) and β_1 in (6), and define

$$\beta_2 \equiv \rho + \delta + \frac{1}{2}\gamma(1-\gamma)\sigma_y^\top \sigma_y - (1-\gamma)\mu_y \tag{26}$$

and

$$\beta_3 \equiv (\gamma\sigma_y - \kappa)^\top(\gamma\sigma_y - \kappa). \tag{27}$$

Then, here is the solution for the NBC (no borrowing constraint) case.

Theorem 2 (NBC) *Suppose $\nu > 0$, $\beta_1 > 0$, $\beta_2 > 0$, and that the borrowing constraint holds with strict inequality at the initial condition:*

$$W_0 > -(1 - R_{0-})\frac{y_0}{\beta_1}. \tag{28}$$

The solution to the investor's Problem 2 can be written in terms of the dual variable x_t (a normalized marginal utility of consumption). Specifically, let the dual variable be defined by

$$x_t \equiv x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t + \sigma_x^\top Z_t} \tag{29}$$

(warning: the same interpretation but not the same process as in Theorem 1), where x_0 solves

$$-y_0 \varphi_x(x_0, R_0) = W_0, \tag{30}$$

where

$$\varphi(x, R) = \begin{cases} -\hat{\eta}\frac{x^b}{b} & \text{if } R = 1 \text{ or } x \le \underline{x} \\ A_+ x^{\alpha_-} - \eta\frac{x^b}{b} + \frac{1}{\beta_1}x & \text{otherwise,} \end{cases} \tag{31}$$

where b, η, and $\hat{\eta}$ were defined in Theorem 1 (in equations (16), (18), and (19)), and

$$A_+ \equiv \frac{1}{\gamma(b-\alpha_-)\beta_1}\underline{x}^{1-\alpha_-}, \tag{32}$$

the optimal retirement boundary is

$$\underline{x} = \left(\frac{(\eta - \hat{\eta})(b - \alpha_-)\beta_1}{b(1-\alpha_-)}\right)^\gamma, \tag{33}$$

where

$$\alpha_- \equiv \frac{\beta_1 - \beta_2 + \frac{1}{2}\beta_3 - \sqrt{(\beta_1 - \beta_2 + \frac{1}{2}\beta_3)^2 + 2\beta_2\beta_3}}{\beta_3}. \tag{34}$$

Then the optimal consumption policy is

$$c_t^* = K^{-bR_t^*} y_t x_t^{-1/\gamma}, \tag{35}$$

the optimal trading strategy is

$$\theta_t^* = y_t[(\sigma^\top\sigma)^{-1}(\mu - r\mathbf{1})x_t\varphi_{xx}(x_t, R_t^*) - \sigma^{-1}\sigma_y(\gamma x_t\varphi_{xx}(x_t, R_t^*) + \varphi_x(x_t, R_t^*))], \tag{36}$$

the optimal bequest policy is

$$B_t^* = k^{-b} y_t x_t^{-1/\gamma}, \tag{37}$$

the optimal retirement policy is

$$R_t^* = \iota\{t \geq \tau^*\}, \tag{38}$$

the corresponding retirement wealth threshold is

$$\overline{W}_t = -y_t\varphi_x(\underline{x}, 0), \tag{39}$$

and the optimal wealth is

$$W_t^* = -y_t\varphi_x(x_t, R_t^*), \tag{40}$$

where

$$\tau^* = (1 - R_0)\inf\{t \geq 0 : x_t \leq \underline{x}\}. \tag{41}$$

Furthermore, the value function is

$$V(W, y, R) = y^{1-\gamma}\left(\varphi(x, R) - x\varphi_x(x, R)\right), \tag{42}$$

where x solves

$$-y\varphi_x(x, R) = W. \tag{43}$$

PROOF. See the proof after Theorem 3.

Theorem 3 (BC)
Suppose $\nu > 0$, $\beta_1 > 0$, $\beta_2 > 0$ and that initial wealth is strictly positive:

$$W_0 > 0. \tag{44}$$

The solution to the investor's Problem 3 is similar to the solution to Problem 2, and can be written in terms of the new dual variable x_t defined by

$$x_t = \frac{x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top\sigma_x)t - \sigma_x^\top Z_t}}{\max(1, \sup_{0\leq s\leq\min(t,\tau^*)} x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top\sigma_x)s - \sigma_x^\top Z_s}/\overline{x})} \tag{45}$$

where x_0 solves

$$-y_0\varphi_x(x_0, R_0) = W_0, \tag{46}$$

and μ_x and μ_y are the same as in Theorems 1 and 2 (as given by (21) and (22)). The new dual value function is

$$\varphi(x,R)=\begin{cases}-\hat{\eta}\frac{x^b}{b} & \textit{if } R=1 \textit{ or } x\leq \underline{x}\\ A_+x^{\alpha_-}+A_-x^{\alpha_+}-\eta\frac{x^b}{b}+\frac{1}{\beta_1}x & \textit{otherwise,}\end{cases} \tag{47}$$

where

$$A_-=\frac{\eta(b-\alpha_-)}{\alpha_+(\alpha_+-\alpha_-)}\overline{x}^{b-\alpha_+}-\frac{1-\alpha_-}{\alpha_+(\alpha_+-\alpha_-)\beta_1}\overline{x}^{1-\alpha_+}, \tag{48}$$

$$A_+=\frac{\eta(\alpha_+-b)}{\alpha_-(\alpha_+-\alpha_-)}\overline{x}^{b-\alpha_-}-\frac{\alpha_+-1}{\alpha_-(\alpha_+-\alpha_-)\beta_1}\overline{x}^{1-\alpha_-}, \tag{49}$$

the x value at which the financial wealth is zero is

$$\overline{x}=\left(\frac{\left(\frac{\eta-\hat{\eta}}{b}\zeta^{b-\alpha_-}-\frac{\eta}{\alpha_-}\right)(\alpha_+-b)\beta_1}{(\zeta^{1-\alpha_-}-\frac{1}{\alpha_-})(\alpha_+-1)}\right)^{\gamma}, \tag{50}$$

the optimal retirement boundary

$$\underline{x}=\zeta\overline{x}, \tag{51}$$

where $\zeta\in(0,1)$ is the unique solution to $q(\zeta)=0$, where

$$\begin{aligned}q(\zeta)\;\equiv\;&\left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\zeta^{b-\alpha_-}-\frac{1}{\alpha_-}\right)\left(\zeta^{1-\alpha_+}-\frac{1}{\alpha_+}\right)(\alpha_+-b)(\alpha_--1)\\ -&\left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\zeta^{b-\alpha_+}-\frac{1}{\alpha_+}\right)\left(\zeta^{1-\alpha_-}-\frac{1}{\alpha_-}\right)(\alpha_--b)(\alpha_+-1),\end{aligned} \tag{52}$$

and

$$\alpha_+=\frac{\beta_1-\beta_2+\frac{1}{2}\beta_3+\sqrt{(\beta_1-\beta_2+\frac{1}{2}\beta_3)^2+2\beta_2\beta_3}}{\beta_3}. \tag{53}$$

Then given the new dual variable x_t and the new dual value function, the rest of the form of the solution are given by (35) through (43).

PROOF OF THEOREMS 2 AND 3: If $R_0=1$, then Problems 2 and 3 are identical to Problem 1. Therefore, the optimality of the claimed optimal strategy follows from Theorem 1. From now on, we assume w.l.o.g. that $R_0=0$. It is tedious but straightforward to use the generalized Itô's lemma, equations (31)-(40), and (47)-(53) to verify that the claimed optimal strategy W_t^*, c_t^*, θ_t^*, and R_t^* in these two theorems satisfy the budget constraint (2). In addition, by Lemmas 4 and 5, x_0 exists and is unique and W_t^* satisfies the borrowing constraint in each problem. Furthermore by Lemma 7, there is a unique solution to (52).

By Doob's optional sampling theorem, we can restrict attention w.l.o.g. to the set of feasible policies that implement the optimal policy stated in Theorem 1 after retirement, and the utility function for such a strategy can be written as

$$E\int_0^{\infty}e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+V(W_s,y_s,1)dR_s\right]. \tag{54}$$

Accordingly, define

$$
\begin{aligned}
M_t &= \int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + V(W_s,y_s,1)dR_s\right] \\
&\quad +(1-R_t)e^{-(\rho+\delta)t}V(W_t,y_t,0). \qquad (55)
\end{aligned}
$$

By Lemma 6 in the Appendix, M_t is a supermartingale for any feasible policy (c,B,R,W) and a martingale for the claimed optimal policy (c^*,B^*,R^*,W^*), which implies that $M_0 \geq E[M_t]$, i.e.,

$$
\begin{aligned}
V(W_0,y_0,0) &\geq E\int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + V(W_s,y_s,1)dR_s\right] \\
&\quad +E[(1-R_t)e^{-(\rho+\delta)t}V(W_t,y_t,0)], \qquad (56)
\end{aligned}
$$

and with equality for the claimed optimal policy. In addition, by Lemma 6, we also have that

$$
\lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}V(W_t,y_t,0)] \geq 0,
$$

with equality for the claimed optimal policy.

Therefore, taking the limit as $t \uparrow \infty$ in (56), we have

$$
V(W_0,y_0,0) \geq E\int_0^\infty e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + V(W_s,y_s,1)dR_s\right],
$$

with equality for the claimed optimal policy (c^*,B^*,R^*). This completes the proof. ♣

We next provide results on computing the human capital at any point in time.

Proposition 1 *Consider the optimal policies stated in Theorems 1-3. After retirement, the market value of the human capital is zero. Before retirement, in Theorem 1 the market value of the human capital is*

$$
H(y_t,t) = g(t)y_t,
$$

where y_t *and* $g(t)$ *are given in (3) and (5); in Theorem 2 the market value of the human capital is*

$$
H(x_t,y_t) = \frac{y_t}{\beta_1}(-\underline{x}^{1-\alpha_-}x_t^{\alpha_- -1}+1),
$$

where y_t, β_1, x_t, $\underline{x}$, α_-, *and* R_t^* *are given in (3), (6), (20), (33), (34), and (38); and in Theorem 3 the market value of the human capital is*

$$
H(x_t,y_t) = \frac{y_t}{\beta_1}(Ax_t^{\alpha_- -1}+Bx_t^{\alpha_+ -1}+1),
$$

where y_t, β_1, α_-, x_t, α_+, *and* R_t^*, *are given in (3), (6), (34), (45), (53), and (38)), and where*

$$
A = \frac{(1-\alpha_+)\underline{x}^{1-\alpha_-}\overline{x}^{\alpha_+-\alpha_-}}{(\alpha_+-1)\overline{x}^{\alpha_+-\alpha_-}-(\alpha_--1)\underline{x}^{\alpha_+-\alpha_-}}
$$

and

$$B = \frac{(\alpha_- - 1)\underline{x}^{1-\alpha_-}}{(\alpha_+ - 1)\overline{x}^{\alpha_+ - \alpha_-} - (\alpha_- - 1)\underline{x}^{\alpha_+ - \alpha_-}}.$$

PROOF OF PROPOSITION 1: See the appendix.

Finally, we can use the results in the following proposition to compute the expected time to retirement.

Proposition 2 *Suppose that the investor is not retired and* $\frac{1}{2}\sigma_x^2 - \mu_x > 0$. *Then the expected time to retirement for the optimal policy is*

$$E_t[\tau^*|x_t = x] = \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2 - \mu_x}, \forall x_t > \underline{x}$$

in Theorem 2 and

$$E_t[\tau^*|x_t = x] = \frac{\underline{x}^m - x^m}{(\frac{1}{2}\sigma_x^2 - \mu_x)m\overline{x}^m} + \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2 - \mu_x}, \forall x_t \in [\underline{x}, \overline{x}] \tag{57}$$

in Theorem 3, where

$$m = 1 - \frac{2\mu_x}{\sigma_x^2}.$$

PROOF OF PROPOSITION 2: See the appendix.

V. Imperfectly Correlated Labor Income

Suppose labor income is not spanned. Specifically, assume

$$(\forall t \geq 0) \quad \frac{dy_t}{y_t} = \mu_y dt + \sigma_y dZ_t + \hat{\sigma}_y d\hat{Z}_t, \tag{58}$$

where $\hat{Z}_t$ is a one-dimensional Brownian motion independent of Z_t. The primal problem is difficult to solve due to a singular boundary condition at $W_t = 0$. We therefore solve this case also using the dual approach.

Let a convex and decreasing function $\varphi(x, R)$ be such that the value function $V(W, y, R) = y^{1-\gamma}(\varphi(x, R) - x\varphi_x(x, R))$, where x solves $-y\varphi_x(x, R) = W$. Then after retirement $\varphi(x, 1)$ is the same as the one in Problems 1-3. After straightforward simplification, the HJB equation for $\varphi(x, 0)$ becomes

$$\frac{1}{2}\beta_3 x^2 \varphi_{xx}(x,0) - (\beta_1 - \beta_2)x\varphi_x(x,0) - \beta_2\varphi(x,0) - \frac{1}{2}\hat{\sigma}_y^2 \frac{\varphi_x^2(x,0)}{\varphi_{xx}(x,0)} - (1+\delta k^{-b})\frac{x^b}{b} + x = 0, \tag{59}$$

where

$$\beta_1 = r + \delta - \mu_y + \sigma_y \kappa^\top + \gamma \hat{\sigma}_y^2, \tag{60}$$

$$\beta_2 = \rho + \delta + \frac{1}{2}\gamma(1-\gamma)(\sigma_y^\top \sigma_y + \hat{\sigma}_y^2) - (1-\gamma)\mu_y, \tag{61}$$

and β_3 is the same as in (27). Note that if the labor income is perfectly correlated with the stock market, i.e., $\hat{\sigma}_y = 0$, then this ODE reduces to (82) for the previous section.

We need to solve ODE (59) subject to (83)-(86). Different from the case with perfectly correlated labor income, the HJB ODE (59) is fully nonlinear and an explicit form for the value function seems unavailable. However, this nonlinear ODE with free boundaries can be easily solved numerically, as we did to produce Figures 7 and 8.

VI. Conclusion

We have constructed workhorse models of lifetime consumption and investment. We hope these models will be useful for analyzing retirement, pensions, and insurance.

Appendix

Here we collect required intermediate results and proofs. First, a lemma

Lemma 1 *Suppose $\nu > 0$, and define $\{R_s\}$ as in Problem 1. Then*

(i) For any t, $g(t)y_t = \frac{1}{\xi_t}E_t[\int_t^\infty \xi_s(1-R_s)y_s\, ds]$. (In Problem 1, $g(t)y_t$ is the market value at t of the subsequent labor income.)

(ii) For any strategy (c, θ, B, W) that is feasible for Problem 1,

$$E\left[\int_0^\infty \xi_s(c_s + \delta B_s)ds\right] \leq W_0 + g(0)y_0, \tag{62}$$

with equality for the claimed optimal strategy $(c^*, \theta^*, B^*, W^*)$ in Theorem 1. (This is the static budget constraint. Inequality for a general policy may be due to investing wealth forever without consuming and/or a suicidal strategy.)

PROOF OF LEMMA 1 (i) By Itô's lemma, (3), (5), (6), (23), and simple algebra,

$$d\left(\xi_t g(t) y_t\right) = -\xi_t(1-R_t)y_t dt + \xi_t g(t) y_t (\sigma_y - \kappa)^\top dZ_t.$$

Furthermore,

$$E\int_0^t \left(\xi_s g(s) y_s\right)^2 (\sigma_y - \kappa)^\top (\sigma_y - \kappa) ds < \infty,$$

since $\xi_s y_s$ is a standard lognormal diffusion and the other factors are bounded for any t and zero for $t > T$. Therefore the local martingale

$$\xi_t g(t) y_t + \int_0^t \xi_s (1 - R_s) y_s \, ds = g(0) y_0 + \int_{s=0}^t \xi_s g(s) y_s (\sigma_y - \kappa)^\top dZ_s \tag{63}$$

is a martingale that is constant for $t > T$. Picking any $\mathcal{T} > \max(t, T)$, the definition of a martingale implies that

$$\xi_t g(t) y_t + \int_0^t \xi_s (1 - R_s) y_s \, ds = E_t \left[\xi_{\mathcal{T}} g(\mathcal{T}) y_{\mathcal{T}} + \int_0^{\mathcal{T}} \xi_s (1 - R_s) y_s \, ds \right] \tag{64}$$

Now, $\mathcal{T} > \max(t, T)$ implies that $g(\mathcal{T}) = 0$, and the integral on the left-hand-side is known at t. Therefore, we can subtract the integral from both sides and divide both sides by ξ_t to conclude

$$\begin{aligned} g(t) y_t &= \frac{1}{\xi_t} E_t \left[\int_t^{\mathcal{T}} \xi_s (1 - R_s) y_s \, ds \right] \\ &= \frac{1}{\xi_t} E_t \left[\int_t^{\infty} \xi_s (1 - R_s) y_s \, ds \right], \end{aligned} \tag{65}$$

where the second equality follows from the fact that $R_s \equiv 1$ for $s > \mathcal{T}$.

(ii) The no-borrowing-without-repayment constraint (4), together with nonnegativity of c_t, B_t, and ξ_t, imply that

$$\xi_t W_t + \int_0^t \xi_s (c_s + \delta B_s - (1 - R_s) y_s) \, ds \geq -[\xi_t g(t) y_t + \int_0^t \xi_s (1 - R_s) y_s \, ds]. \tag{66}$$

For any strategy (c, θ, B, W) that is feasible in Problem 1, Itô's lemma, the budget constraint (2), the definition of g, (23), and (7) imply that the left-hand side of (66) has zero drift and is therefore a local martingale. Furthermore, the right-hand side of (66) is a martingale, which was an intermediate result that (63) is a martingale in the proof of Part (i) above. Since any local martingale bounded below by a martingale is a supermartingale,[4] the left-hand-side of (66) is a supermartingale for any feasible strategy.

For the claimed optimal strategy, the left-hand side of (66) is a martingale, since it is a local martingale and the integrand with respect to dZ_t is the sum of lognormal terms that can be shown to be bounded in L^2 over finite time intervals.

By definition of supermartingale and martingale, we have for any $t > 0$ that

$$W_0 \geq E \left[\xi_t W_t + \int_0^t \xi_s (c_s + \delta B_s - (1 - R_s) y_s) \, ds \right], \tag{67}$$

with equality for the claimed optimum. Taking the limit as $t \uparrow \infty$ and using the result from (i), we have that

$$W_0 \geq \liminf_{t \uparrow \infty} E[\xi_t W_t] + E \left[\int_0^{\infty} \xi_s (c_s + \delta B_s) \, ds \right] - g(0) y_0. \tag{68}$$

[4] If a local martingale X is bounded below by a martingale Y, then $X - Y$ is a local martingale that is bounded below and is therefore a supermartingale. Therefore, $X = (X - Y) + Y$ is a supermartingale.

Since $g(t) = 0$ for $t > T$, the no-borrowing-without-repayment constraint (4) implies that $\liminf_{t\uparrow\infty} E[\xi_t W_t] \geq 0$ for any feasible strategy. Furthermore, $g(t) = 0$ for $t > T$ also implies that the claimed optimal wealth W_t^* is lognormal for $t > T$ and it is straightforward to verify that $\nu > 0$ implies that $\lim_{t\uparrow\infty} E[\xi_t W_t^*] = 0$. Therefore,

$$W_0 \geq E\left[\int_0^\infty \xi_s(c_s + \delta B_s)\, ds\right] - g(0)y_0, \tag{69}$$

with equality for the claimed optimum, which can be rewritten as what is to be shown. ♣

Lemma 2 *(i) Let y_t, x_t, ξ_t be as defined in (3), (20), and (23) respectively. Then*

$$\xi_t = e^{-(\rho+\delta)t}\left(\frac{y_t}{y_0}\right)^{-\gamma}\left(\frac{x_t}{x_0}\right). \tag{70}$$

(ii) Given any feasible strategy (c, B) for Problem 1 and the claimed optimal strategy (c^*, B^*) in Theorem 1, we have

$$\begin{aligned} & E[\int_0^\infty e^{-(\rho+\delta)t}\left(\frac{(K^{R_t}c_t)^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_t)^{1-\gamma}}{1-\gamma}\right)dt] \\ \leq \;\; & E[\int_0^\infty e^{-(\rho+\delta)t}\left(\frac{(K^{R_t}c_t^*)^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_t^*)^{1-\gamma}}{1-\gamma}\right)dt] \\ & +\frac{x_0}{y_0^\gamma}E[\int_0^\infty \xi_t(c_t + \delta B_t - c_t^* - \delta B_t^*)dt]. \end{aligned} \tag{71}$$

PROOF OF LEMMA 2: (i) This can be directly verified using (23), (20), (21), (22), and (3)

(ii) Because any concave function lies below the tangent line at any point, we have that for any positive c, c^*, B, and B^*, we have

$$\frac{(K^R c)^{1-\gamma}}{1-\gamma} \leq \frac{(K^R c^*)^{1-\gamma}}{1-\gamma} + (K^R)^{1-\gamma}c^{*-\gamma}(c - c^*) \tag{72}$$

and

$$\frac{(kB)^{1-\gamma}}{1-\gamma} \leq \frac{(kB^*)^{1-\gamma}}{1-\gamma} + k^{1-\gamma}B^{*-\gamma}(B - B^*). \tag{73}$$

Then (71) follows from (13), (15), and (70). ♣

Lemma 3 *(Dominated Convergence Theorem) Suppose that a.s.-convergent sequences of random variables $X_n \to X$ and $Y_n \to Y$ satisfy $0 \leq X_n \leq Y_n$ and $E[Y_n] \to E[Y] < \infty$. Then $E[X_n] \to E[X]$.*

PROOF OF LEMMA 3: Since $0 \leq X_n \leq Y_n$, by Fatou's lemma $\liminf E[X_n] \geq E[X]$ and $\liminf E[Y_n - X_n] \geq E[Y - X]$. These inequalities imply that both $\limsup E[X_n] \geq E[X]$ and $\liminf E[X_n] \leq E[X]$ since $E[Y_n] \to E[Y] < \infty$. Therefore, we must have $E[X_n] \to E[X]$. ♣

Next, let

$$\psi(x) \equiv A_+ x^{\alpha_-} - \eta \frac{x^b}{b} + \frac{1}{\beta_1} x \tag{74}$$

and

$$\hat{\psi}(x) \equiv -\hat{\eta} \frac{x^b}{b}, \tag{75}$$

where A_+ is as defined in Theorem 2.

Lemma 4 *In Theorem 2, suppose $\nu > 0$, $\beta_1 > 0$, and $\beta_2 > 0$. Then*

(i). $\hat{\psi}(x)$ *is strictly decreasing and strictly convex;*

(ii). $\psi(x)$ *is strictly convex and* $\psi_x(x) \leq \frac{1}{\beta_1}$*;*

(iii). $\forall x \geq 0$, *we have* $\psi(x) \geq \hat{\psi}(x)$ *and* $\forall x \geq \underline{x}$, *we have* $\psi_x(x) \geq \hat{\psi}_x(x)$.

(iv). Given (28), there exists a unique solution $x_0 > 0$ to (30). In addition, W_t^* defined in (40) satisfies the borrowing constraint (8).

PROOF OF LEMMA 4: (i). This follows from direct differentiation. (ii). These results also follow from direct differentiation, noting that $A_+ > 0$ and $\alpha_- < 0$. (iii). This follow from a similar argument to that for Part (ii) of Lemma 5 below. (iv). By (31), (74), and (75), we have

$$\varphi(x, R) = \begin{cases} \hat{\psi}(x) & \text{if } R = 1 \text{ or } x \leq \underline{x} \\ \psi(x) & \text{otherwise.} \end{cases} \tag{76}$$

By Part (i), Part (ii), and $\psi_x(\underline{x}) = \hat{\psi}_x(\underline{x})$, $\varphi'(x)$ is continuous and strictly increasing in x. By inspection of (74) and (75), $\varphi_x(x, R)$ takes on all values that are less than or equal to $\frac{1}{\beta_1}$. Since $y_0 > 0$, there exists a unique solution $x_0 > 0$ to (30) for each $W_0 \geq -\frac{1}{\beta_1}$. Also, since $\varphi_x(x, R) \leq \frac{1}{\beta_1}$, (40) implies that $W_t^* > -(1 - R_t)\frac{y_t}{\beta_1}, \forall t \geq 0$. ♣

Next, let

$$\psi(x) \equiv A_+ x^{\alpha_-} + A_- x^{\alpha_+} - \eta \frac{x^b}{b} + \frac{1}{\beta_1} x \tag{77}$$

and

$$\hat{\psi}(x) \equiv -\hat{\eta} \frac{x^b}{b}, \tag{78}$$

where A_+ and A_- are as defined in Theorem 3.

Lemma 5 *In Problem 3, suppose $\nu > 0$, $\beta_1 > 0$, and $\beta_2 > 0$. Suppose there exists a solution $\zeta \in (0,1)$ to equation (52). Then*

(i). $\hat{\psi}(x)$ *is strictly convex and strictly decreasing for $x \geq 0$;*

(ii). $\forall x \leq \overline{x}$ *we have* $\psi(x) \geq \hat{\psi}(x)$*,* $\forall x \in [\underline{x}, \overline{x}]$ *we have* $\psi_x(x) \geq \hat{\psi}_x(x)$ *and*

$$\underline{x} < \left(\frac{1-K^{-b}}{b}\right)^{\gamma}. \tag{79}$$

(iii).

$$A_- < 0, \;\; A_+ > 0, \;\; \overline{x} > \left(\frac{(1-\alpha_-)(1+\delta k^{-b})}{b-\alpha_-}\right)^{\gamma}.$$

(iv). $\psi(x)$ *is strictly convex and strictly decreasing for $x < \overline{x}$.*

(v). Given $W_0 > 0$, there exists a unique solution $x_0 > 0$ to (46). In addition, W_t^* defined in (40) satisfies the borrowing constraint (9).

PROOF OF LEMMA 5: (i). $\gamma > 0$ implies that $b = 1 - 1/\gamma < 1$. Then since $\nu > 0$, direct differentiation shows that $\hat{\psi}(x)$ is strictly convex and strictly decreasing for $x > 0$.

(ii). First, since $\nu > 0$, $\beta_1 > 0$, and $\beta_2 > 0$, it is straightforward to use the definitions of α_+ and α_- show that

$$\alpha_+ > 1 > b > \alpha_-, \;\; \alpha_- < 0. \tag{80}$$

Recall the definitions of $\psi(x)$ and $\hat{\psi}(x)$ in (77) and (78). Let

$$h(x) \equiv \psi(x) - \hat{\psi}(x).$$

It can be easily verified that

$$\frac{1}{2}\beta_3 x^2 \hat{\psi}_{xx}(x) - (\beta_1 - \beta_2)x\hat{\psi}_x(x) - \beta_2\hat{\psi}(x) - (K^{-b} + \delta k^{-b})\frac{x^b}{b} = 0, \tag{81}$$

and

$$\frac{1}{2}\beta_3 x^2 \psi_{xx}(x) - (\beta_1 - \beta_2)x\psi_x(x) - \beta_2\psi(x) - (1 + \delta k^{-b})\frac{x^b}{b} + x = 0, \tag{82}$$

with

$$\psi(\underline{x}) = \hat{\psi}(\underline{x}), \tag{83}$$

$$\psi_x(\underline{x}) = \hat{\psi}_x(\underline{x}), \tag{84}$$

$$\psi_x(\overline{x}) = 0 \tag{85}$$

and

$$\psi_{xx}(\overline{x}) = 0. \tag{86}$$

Then by (81) and (82), $h(x)$ must satisfy

$$\frac{1}{2}\beta_3 x^2 h'' - (\beta_1 - \beta_2)xh' - \beta_2 h = \frac{1-K^{-b}}{b}x^b - x. \tag{87}$$

By (83)-(85) and the fact that $\hat{\psi}(x)$ is monotonically decreasing for $x > 0$, we have

$$h(\underline{x}) = 0, \;\; h'(\underline{x}) = 0, \;\; h'(\overline{x}) > 0. \tag{88}$$

Differentiating (87) once, we obtain

$$\frac{1}{2}\beta_3 x^2 h''' + (\beta_3 - \beta_1 + \beta_2)xh'' - \beta_1 h' = (1-K^{-b})x^{b-1} - 1. \tag{89}$$

We consider two possible cases.

Case 1: $(1-K^{-b})\underline{x}^{b-1} - 1 < 0$. In this case, the RHS of equation (89) is negative. Since $\beta_1 > 0$, $h'(x)$ cannot have any interior nonpositive minimum. To see this, suppose $\hat{x} \in (\underline{x}, \overline{x})$ achieves an interior minimum with $h'(\hat{x}) \leq 0$. Then we would have $h'''(\hat{x}) > 0$ and $h''(\hat{x}) = 0$, which implies that the LHS is positive. A contradiction. Since $h'(\underline{x}) = 0, \;\; h'(\overline{x}) > 0$, we must have $h'(x) > 0$ for any $x \in (\underline{x}, \overline{x}]$ because otherwise there would be an interior nonnegative minimum. Then the fact that $h(\underline{x}) = 0$ implies that $h(x) > 0$ for any $x \in (\underline{x}, \overline{x}]$. Since $h'(x) > 0$ for any $x \in (\underline{x}, \overline{x}]$ and $h'(\underline{x}) = 0$, we must have $h''(\underline{x}) \geq 0$. In addition, if $h''(\underline{x})$ were equal to 0, then we would have $h'''(\underline{x}) < 0$ by (88) and (89) since $(1-K^{-b})\underline{x}^{b-1} - 1 < 0$. However, this would contradict the fact that $h'(x) > 0$ for any $x \in (\underline{x}, \overline{x}]$ and $h'(\underline{x}) = 0$. Therefore we must have $h''(\underline{x}) > 0$. Then (87), (88) and $h''(\underline{x}) > 0$ imply that

$$\underline{x} < \left(\frac{1-K^{-b}}{b}\right)^{\gamma},$$

Case 2: $(1-K^{-b})\underline{x}^{b-1} - 1 \geq 0$. In this case, we must have $0 < b < 1$ because $K > 1$. Therefore $\underline{x} \leq \left(1-K^{-b}\right)^{\gamma} < \left(\frac{1-K^{-b}}{b}\right)^{\gamma}$. This implies that $h''(\underline{x}) > 0$ by (87) and (88). Therefore there exists $\epsilon > 0$ such that $h'(x) > 0$ for any $x \in (\underline{x}, \underline{x}+\epsilon]$ because $h'(\underline{x}) = 0$. The RHS of equation (89) is monotonically decreasing in x. Let x^* be such that the RHS of (89) is 0. Then for any $x \leq x^*$, the RHS is nonnegative and thus $h'(x)$ cannot have any interior nonnegative (local) maximum in $[\underline{x}, x^*]$ for similar reasons to those in Case 1. Thus there cannot exist any $\hat{x} \in (\underline{x}+\epsilon, x^*]$ such that $h'(\hat{x}) \leq 0$. If $x^* < \overline{x}$, then for any $x \in (x^*, \overline{x}]$, the RHS is nonpositive and thus $h'(x)$ cannot have any interior nonpositive (local) minimum in $(x^*, \overline{x}]$. Thus there cannot exist any $\hat{x} \in (x^*, \overline{x}]$ such that $h'(\hat{x}) \leq 0$. Therefore, there cannot exist any $\hat{x} \in (\underline{x}, \overline{x})$ such that $h'(\hat{x}) \leq 0$ and thus we have $h'(x) > 0$ and $h(x) > 0$ for any $x \in (\underline{x}, \overline{x}]$.

Now we show for both cases, $h(x) > 0$ for any $x < \underline{x}$. (79) implies that the RHS of (87) is positive for $x < \underline{x}$ and h cannot achieve an interior positive maximum for $x < \underline{x}$. On the other hand, $h''(\underline{x}) > 0$, $h''(x)$ is continuous at $\underline{x}$, and $h'(\underline{x}) = 0$ imply that there exists an $\epsilon > 0$ such that

$$\forall\, x \in [\underline{x}-\epsilon, \underline{x}], \;\; h'(x) < 0. \tag{90}$$

Thus $\forall x \in [\underline{x} - \epsilon, \underline{x})$, $h(x) > 0$. Therefore $\forall x < \underline{x}$, $h(x) > 0$, since otherwise h would achieve an interior positive maximum in $(0, \underline{x})$.

(iii). It can be shown that

$$A_+ = \frac{(\eta - \hat{\eta})(\alpha_+ - b)}{b(\alpha_+ - \alpha_-)} \underline{x}^{b-\alpha_-} - \frac{(\alpha_+ - 1)}{(\alpha_+ - \alpha_-)\beta_1} \underline{x}^{1-\alpha_-}$$

and

$$\eta = \frac{(\alpha_+ - 1)(1 - \alpha_-)(1 + \delta k^{-b})}{(\alpha_+ - b)(b - \alpha_-)\beta_1}. \tag{91}$$

(79) then implies that $A_+ > 0$. Since we also have (49), $\overline{x}$ must satisfy

$$\overline{x} > \left(\frac{\eta(\alpha_+ - b)\beta_1}{\alpha_+ - 1} \right)^{\gamma}.$$

Since

$$\frac{\alpha_+ - b}{\alpha_+ - 1} > \frac{b - \alpha_-}{1 - \alpha_-},$$

we have

$$\overline{x} > \left(\frac{\eta(b - \alpha_-)\beta_1}{1 - \alpha_-} \right)^{\gamma},$$

which (by the definition (48)) implies that $A_- < 0$.

(iv). Differentiating (77) twice, we have, for $x < \overline{x}$,

$$\begin{aligned} \psi_{xx}(x) &= (A_- \alpha_+(\alpha_+ - 1)x^{\alpha_+ - b} + A_+ \alpha_-(\alpha_- - 1)x^{\alpha_- - b} - \eta(b-1))x^{b-2} \\ &> \psi_{xx}(\overline{x})(x/\overline{x})^{b-2} = 0, \end{aligned} \tag{92}$$

where the inequality follows from the fact that

$$\frac{d}{dx}[A_- \alpha_+(\alpha_+ - 1)x^{\alpha_+ - b} + A_+ \alpha_-(\alpha_- - 1)x^{\alpha_- - b}] < 0,$$

which is implied by $A_- < 0$, $A_+ > 0$ and (80), and the last equality in (92) follows from $\psi_{xx}(\overline{x}) = 0$. Thus $\psi(x)$ is strictly convex $\forall x < \overline{x}$. Since $\psi_x(\overline{x}) = 0$ and $\forall x < \overline{x}$, $\psi_{xx}(x) > 0$, we must also have $\forall x < \overline{x}$, $\psi_x(x) < 0$.

(v). By (47), (77), and (78), we have

$$\varphi(x, R) = \begin{cases} \hat{\psi}(x) & \text{if } R = 1 \text{ or } x \le \underline{x} \\ \psi(x) & \text{otherwise.} \end{cases} \tag{93}$$

By Part (i), Part (iv), and $\psi_x(\underline{x}) = \hat{\psi}_x(\underline{x})$, $\varphi'(x)$ is continuous and strictly increasing in x. By inspection of (77) and (78), $\varphi_x(x, R)$ takes on all nonpositive values. Since $y_0 > 0$, there exists a unique solution $x_0 > 0$ to (46) for each $W_0 > 0$. Also, since $\varphi_x(x, R) < 0$, (40) implies that $W_t^* > 0, \forall t \ge 0$. ♣

Lemma 6 *Given the definitions in Theorem 2 and Theorem 3,*

1. *M_t as defined in (55) is a supermartingale for any feasible policy and a martingale for the claimed optimal policy.*

2.
$$\lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}(1-\gamma)V(W_t, y_t, 0)] \geq 0, \tag{94}$$
with equality for the claimed optimal policy.

PROOF OF LEMMA 6:

(i) Define $\overline{W} = -y\varphi_x(\underline{x}, 0)$. Then for any $W \geq 0$,

$$V(W, y, 0) \geq V(W, y, 1), \tag{95}$$

with equality for $W \geq \overline{W}$.[5]

Applying the generalized Itô's lemma to M_t defined in the proofs of Theorems 2 and 3, we have

$$\begin{aligned} M_t &= M_0 + \int_0^t (1-R_s)\left\{e^{-(\rho+\delta)s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right) + E\left[d\Big(e^{-(\rho+\delta)s}V(W_s, y_s, 0)\Big)\right]\right\} ds \\ &\quad + \int_0^t e^{-(\rho+\delta)s}(V(W_s, y_s, 1) - V(W_s, y_s, 0))dR_s \\ &\quad + \int_0^t (1-R_s)e^{-(\rho+\delta)s}(V_W(W_s, y_s, 0)\theta_s^\top\sigma + y_sV_y(W_s, y_s, 0)\sigma_y^\top)dZ_s, \end{aligned} \tag{96}$$

By the definitions of V, φ, (c^*, B^*, R^*, W^*), and the fact that $\varphi(x, 0)$ satisfies (81)-(84), we obtain that the first integral is always nonpositive for any feasible policy (c, B, θ, R) and is equal to zero for the claimed optimal policy $(c^*, B^*, \theta^*, R^*)$. By (95), the third term in (96) is always nonpositive for every feasible retirement policy R_t and equal to zero for the claimed optimal policy R_t^*. In addition, using the expressions for the claimed optimal θ_t^*, V, B_t^*, and W_t^*, we have that under the claimed optimal policy, the stochastic integral is a martingale because (1) y_t is a geometric Brownian motion; (2) in Theorem 2, x_t is also a geometric Brownian motion; (3) in Theorem 3, x_t is bounded between $\underline{x}$ and $\overline{x}$ before retirement, and x_t is also a geometric Brownian motion after retirement. This shows that M_t is a local supermartingale for all feasible policies and a martingale for the claimed optimal policy.

If $\gamma < 1$, then $V(W_t, y_t, 0) > 0$ and thus

$$\lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}(1-\gamma)V(W_t, y_t, 0)] \geq 0.$$

[5]This can be shown as follows: Let x and x^R be such that $-y\varphi_x(x, 0) = W$ and $-y\varphi_x(x^R, 1) = W$. Then we have

$$\varphi(x, 0) - \varphi(x^R, 1) \geq \varphi(x, 1) - \varphi(x^R, 1) \geq \varphi_x(x^R, 1)(x - x^R) = x\varphi_x(x, 0) - x^R\varphi_x(x^R, 1),$$

where the first inequality follows from $\varphi(x, 0) \geq \varphi(x, 1)$ by Lemmas 4 and 5 and the second inequality from the convexity of $\varphi(x, 1)$. After rearranging, we obtain (95).

In addition, M_t is always nonnegative and thus a supermartingale.

If $\gamma > 1$, we divide the proof that M_t is actually a supermartingale for any feasible policy into two parts: One for Theorem 2 and the other for Theorem 3.

(A) For Theorem 2, consider an investor who has an initial endowment of $(W_0, (1+\epsilon)y_0)$ ($\epsilon > 0$) but follows the same strategy (c, B, θ, R) for an investor who has an initial endowment of (W_0, y_0) and saves the additional income until retirement, and follows the optimal strategy given the implied wealth afterwards. Let the implied wealth process be W_t^ϵ, which converges to W_t as $\epsilon \to 0$. By (96), there exists a series of increasing stopping times $\tau_n \to \infty$ such that

$$\begin{aligned} & V(W_0, (1+\epsilon)y_0, 0) \\ \geq\ & E\int_0^{\tau_n \wedge t} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + V(W_s^\epsilon, (1+\epsilon)y_s, 1)dR_s\right] \\ & +E[(1-R_{\tau_n\wedge t})e^{-(\rho+\delta)(\tau_n\wedge t)}V(W^\epsilon_{\tau_n\wedge t}, (1+\epsilon)y_{\tau_n\wedge t}, 0)]. \end{aligned} \tag{97}$$

Since the integrand in the integral of (97) is always negative, this integral is monotonically decreasing in time. In addition,

$$\begin{aligned} 0 \ &\geq\ (1-R_t)e^{-(\rho+\delta)t}V(W_t^\epsilon, (1+\epsilon)y_t, 0) \\ &\geq\ e^{-(\rho+\delta)t}V(-\frac{y_t}{\beta_1}, (1+\epsilon)y_t, 0) \\ &=\ V(-\frac{1}{\beta_1}, (1+\epsilon), 0)e^{-(\rho+\delta)t}y_t^{1-\gamma} & (98) \\ &\geq\ V(-\frac{1}{\beta_1}, (1+\epsilon), 0)N_t, & (99) \end{aligned}$$

where

$$N_t \equiv e^{-\frac{1}{2}(1-\gamma)^2\sigma_y^2 t + (1-\gamma)\sigma_y Z_t} \tag{100}$$

is a martingale with $E[N_t] = 1$, the second inequality follows from V negative and increasing in W and $W_t^\epsilon > W_t > -\frac{y_t}{\beta_1}$, the equality follows from the form of V as defined by (42) and (43), and the last inequality follows from $V(-\frac{1}{\beta_1}, (1+\epsilon), 0) < 0$ and $\beta_2 > 0$. In addition, $V(-\frac{1}{\beta_1}, (1+\epsilon), 0) > -\infty$.

Therefore, taking $n \to \infty$ in (97), by the monotone convergence theorem for the first term and Lemma 3 (a generalized dominated convergence theorem) for the second term, we have

$$\begin{aligned} & V(W_0, (1+\epsilon)y_0, 0) \\ \geq\ & E\int_0^{t} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + V(W_s^\epsilon, (1+\epsilon)y_s, 1)dR_s\right] \\ & +E[(1-R_t)e^{-(\rho+\delta)t}V(W^\epsilon_t, (1+\epsilon)y_t, 0)]. \end{aligned} \tag{101}$$

Next, taking $\epsilon \to 0$ in (101), we obtain $M_0 \geq E[M_t]$ for any $t \geq 0$. Since the above argument applies to any time $s \leq t$, we have $M_s \geq E[M_t]$ for any $t \geq s$ and thus M_t is a supermartingale for every feasible policy.

(B) For Theorem 3, by (96), there exists an increasing sequence of stopping times $\tau_n \to \infty$ such that $M_0 \geq E[M_{\tau_n \wedge t}]$, i.e.,

$$\begin{aligned} & V(W_0, y_0, 0) \\ \geq \;& E\int_s^{\tau_n \wedge t} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + V(W_s, y_s, 1)dR_s\right] \\ & + E[(1-R_{\tau_n\wedge t})e^{-(\rho+\delta)(\tau_n\wedge t)}V(W_{\tau_n\wedge t}, y_{\tau_n \wedge t}, 0)]. \end{aligned} \tag{102}$$

Since the integrand in the integral of (102) is always negative, this integral is monotonically decreasing in time. In addition,

$$\begin{aligned} 0 &\geq (1-R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0) \\ &\geq e^{-(\rho+\delta)t}V(0, y_t, 0) \\ &= V(0,1,0)e^{-(\rho+\delta)t}{y_t}^{1-\gamma} && (103) \\ &\geq V(0,1,0)N_t, && (104) \end{aligned}$$

where N_t, as defined in (100), is a martingale with $E[N_t] = 1$, the second inequality follows from V negative and increasing in W and $W_t > 0$, the equality follows from the form of V as defined by (42) and (43), and the last inequality follows from $V(0,1,0) < 0$ and $\beta_2 > 0$. In addition, $V(0,1,0) > -\infty$.

Therefore, taking $n \to \infty$ in (102), by the monotone convergence theorem for the first term and Lemma 3 (a generalized dominated convergence theorem) for the second term, we have

$$\begin{aligned} & V(W_0, y_0, 0) \\ \geq \;& E\int_0^{t} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + V(W_s, y_s, 1)dR_s\right] \\ & + E[(1-R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0)]. \end{aligned} \tag{105}$$

That is: $M_0 \geq E[M_t]$ for any $t \geq 0$. Since the above argument applies to any time $s \leq t$, we have $M_s \geq E[M_t]$ for any $t \geq s$ and thus M_t is a supermartingale for all feasible policies.

(ii) Since $(1-\gamma)V(W, y, 0) \geq 0$ for every feasible policy,[6] we have

$$\begin{aligned} 0 \;\leq\; & \lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}(1-\gamma)V(W_t, y_t, 0)] \\ & = \lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}y_t^{1-\gamma}(1-\gamma)(\varphi(x_t, 0) - x_t\varphi_x(x_t, 0))] \\ & \leq \lim_{t\to\infty} E[L_1 e^{-(\rho+\delta)t}y_t^{1-\gamma} + \hat{\eta}e^{-(\rho+\delta)t/\gamma}\xi_t^b] \\ & = 0, \end{aligned} \tag{106}$$

[6]This can be shown as follows: $V_W(W, y, 0) = y^{-\gamma}x > 0$ and thus $V(W, y, 0)$ increases in W. If $\gamma < 1$, then $V(W, y, 0) \geq 0$ because $V(W, y, 0) \geq V(W, y, 1) \geq 0$. If $\gamma > 1$, then $V(W, y, 0) < 0$ because $V(W, y, 0) \leq V(\overline{W}, y, 0) = V(\overline{W}, y, 1) < 0$.

where the second inequality follows from (1) the definition of φ in Theorem 2, $A_+ > 0$, $\alpha_- < 0$, and $x_t > \underline{x}$; and (2) the fact that in Theorem 3, x_t, $\varphi(x_t)$, and $\varphi_x(x_t)$ are all bounded, while $R_t = 1$ for $t > \tau^*$. The last equality in (106) follows from the conditions that $\nu > 0$ and $\beta_2 > 0$.

Therefore, for the claimed optimal policy, we obtain

$$\lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0)] = 0.$$

For any feasible policy, if $\gamma < 1$, then $V(W, y, 0) > 0$ and therefore (94) holds. If $\gamma > 1$, since $\beta_2 > 0$, we have $\lim_{t\to\infty} E[e^{-(\rho+\delta)t}{y_t}^{1-\gamma}] = 0$. Therefore, taking the limit as $t \to \infty$ in (98),

$$\lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}V(W_t, (1+\epsilon)y_t, 0)] = 0,$$

which implies that (94) holds, after taking the limit as $\epsilon \to 0$. Similarly taking the limit as $t \to \infty$ in (103), we have that (94) also holds. This completes the proof. ♣

Lemma 7 *Suppose $\nu > 0$, $\beta_1 > 0$, and $\beta_2 > 0$. Then there exists a unique solution $\zeta^* \in (0,1)$ to equation (52) and*

$$\zeta^* < \overline{\zeta} = \textit{Min}(\left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\right)^{\gamma}, 1).$$

PROOF OF LEMMA 7: Since $\nu > 0$, $\beta_1 > 0$, and $\beta_2 > 0$,

$$\alpha_+ > 1 > b > \alpha_-, \quad \alpha_- < 0.$$

Next, since $\zeta^{b-\alpha_+}$ dominates $\zeta^{1-\alpha_+}$ as $\zeta \to 0$, we have

$$\lim_{\zeta\to 0} q(\zeta) = \lim_{\zeta\to 0} -\frac{1-K^{-b}}{b(1+\delta k^{-b})}(\alpha_- - b)(\alpha_+ - 1)\zeta^{b-\alpha_+} = +\infty.$$

Next, it is easy to verify that

$$q(1) = -\frac{(\alpha_+ - 1)(\alpha_- - 1)(\alpha_+ - \alpha_-)(K^{-b} + \delta k^{-b})}{\alpha_+\alpha_-(1+\delta k^{-b})} < 0.$$

Now suppose $\hat{\zeta} = \left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\right)^{\gamma} < 1$. Then we have $\frac{1-K^{-b}}{b(1+\delta k^{-b})}\hat{\zeta}^{b-\alpha_+} - \frac{1}{\alpha_+} = \hat{\zeta}^{1-\alpha_+} - \frac{1}{\alpha_+}$, $\frac{1-K^{-b}}{b(1+\delta k^{-b})}\hat{\zeta}^{b-\alpha_-} - \frac{1}{\alpha_-} = \hat{\zeta}^{1-\alpha_-} - \frac{1}{\alpha_-}$, and $\hat{\zeta}^{1-\alpha_+} > 1 > \frac{1}{\alpha_+}$. It follows that

$$q(\hat{\zeta}) = -\frac{1}{\gamma}(\hat{\zeta}^{1-\alpha_+} - \frac{1}{\alpha_+})(\hat{\zeta}^{1-\alpha_-} - \frac{1}{\alpha_-})(\alpha_+ - \alpha_-) < 0.$$

Then by continuity of q, there exists a solution $\zeta^* \in (0, \overline{\zeta})$ such that $q(\zeta^*) = 0$. Suppose there exists another solution $\hat{\zeta} \in [0, 1)$ such that $q(\hat{\zeta}) = 0$. Let $V(W, y, 0)$ and $\overline{W}$

be the value function and boundary respectively corresponding to ζ^* and $\hat{v}(W, y, 0)$ and $\hat{\overline{W}}$ be the value function and boundary respectively corresponding to $\hat{\zeta}$. Without loss of generality, suppose $\overline{W} > \hat{\overline{W}}$. Since $\hat{\overline{W}}$ is the retirement boundary, the value function corresponding to $\hat{\zeta}$ for $\overline{W} > W > \hat{\overline{W}}$ is equal to $V(W, y, 1)$. However, Lemma 5 implies that $V(W, y, 0) > V(W, y, 1)$ for any $W < \overline{W}$. This implies that $\hat{\overline{W}}$ cannot be the optimal retirement boundary which contradicts Theorem 3. Therefore the solution to equation (52) is unique. ♣

PROOF OF PROPOSITION 1. The result on the market value of human capital for Problem 1 is directly implied by Lemma 1, Part (i).

For the cases with voluntary retirement, since there is no more labor income after retirement, the market value of human capital after retirement is zero. We next prove the claims for after retirement. Using the expressions of H and the dynamics of x_t and y_t, it can be verified that for $x_t > \underline{x}$ in Theorem 2 and for $\underline{x} < x_t < \overline{x}$ in Theorem 3 we have that the change in the market value of human capital plus the flow of labor income is given by

$$\begin{aligned} & d(\xi_t H(x_t, y_t)) + \xi_t y_t dt \\ = \; & \xi_t(\frac{1}{2}\beta_3 x_t^2 H_{xx} - (\beta_1 - \beta_2 - \beta_3) x_t H_x - \beta_1 H + y_t) dt \\ & + \xi_t (x_t H_x \sigma_x^\top + H(\sigma_y^\top - \kappa^\top)) dZ_t. \end{aligned} \tag{107}$$

The drift term in (107) is equal to zero after plugging in the expressions for H (for Theorem 3, the additional local time term at $\overline{x}$ from applying the generalized Itô's lemma is also equal to zero because it can be verified that $H_x(\overline{x}, y_t) = 0$). This implies that

$$\mathcal{M}_t \equiv \xi_t H(x_t, y_t) + \int_0^t \xi_s y_s ds$$

is a local martingale. In addition, there exists a constant $0 < L < \infty$ such that

$$|\xi_t (x_t H_x \sigma_x^\top + H(\sigma_y^\top - \kappa^\top))| < L \xi_t y_t$$

in Theorem 2 since $\alpha_- < 0$ and $x_t > \underline{x}$, and in Theorem 3 since $\underline{x} < x_t \leq \overline{x}$. Since both ξ_t and y_t are geometric Brownian motions, we have that $\mathcal{M}_t$ is actually a martingale. Recall the definition (41) of the optimal retirement time τ^*. We have, $\forall t \leq \tau^*$

$$\xi_t H(x_t, y_t) + \int_0^t \xi_s y_s ds = E_t[\xi_{\tau^*} H(\underline{x}, y_{\tau^*}) + \int_0^{\tau^*} \xi_s y_s ds],$$

which implies that

$$H(x_t, y_t) = \xi_t^{-1} E_t[\int_t^{\tau^*} \xi_s y_s ds],$$

since it can be easily verified that $H(\underline{x}, y) = 0$. Therefore H as specified in the proposition is indeed the market value of the future labor income. ♣

PROOF OF PROPOSITION 2. First we prove the result for Theorem 2. Recall that

$$dx_t = \mu_x x_t dt + \sigma_x x_t dB_t.$$

Let

$$f(x) \equiv \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2 - \mu_x}.$$

Then by Itô's lemma, for any stopping time $\mathcal{T} \geq t$ we have

$$f(x_{\mathcal{T}})+\int_t^{\mathcal{T}} 1ds = f(x_t)+\int_t^{\mathcal{T}} \left(\frac{1}{2}\sigma_x^2 x_s^2 f_{xx} + \mu_x x_s f_x + 1\right) ds+\int_t^{\mathcal{T}} \frac{\sigma_x}{\frac{1}{2}\sigma_x^2 - \mu_x} dZ_s, \quad (108)$$

which implies that $f(x_{\mathcal{T}})+\int_t^{\mathcal{T}} 1ds$ is a martingale since it can be easily verified that the drift term is zero given the definition of $f(x)$ and the stochastic integral is a scaled Brownian motion and thus a martingale. Thus taking $\mathcal{T} = \tau^*$ and taking expectation in (108), we get

$$f(x) = E_t[\tau^* | x_t = x],$$

since $x_{\tau^*} = \underline{x}$ and $f(\underline{x}) = 0$. A similar argument applies to the case for Theorem 3, noting that when evaluated at $x = \overline{x}$, the first derivative of the right hand side of (57) with respect to x is zero and x_t is bounded. This completes the proof. ♣

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