Lifetime Consumption and Investment

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Goal of the research

We build workhorse models of lifetime consumption and investment for answering questions about retirement, pensions, and insurance.

- different consumption preferences before and after retirement
- possibly stochastic wages over time
- voluntary or mandatory retirement date
- nonnegative wealth constraint on borrowing
- idealized hedging using possible life insurance
- bequest motive or not
- wages and mortality can vary systematically through life
- state of health and preferences not just a function of age

Some related literature

Retirement decision determinants

- Mitchell and Fields (1984), Gustman and Steinmeier (2001), Khitatrakun (2002)

Lifetime consumption and investment with

- exogenous income: Jagannathan and Kocherlakota (1996), Cuoco (1997)
- endogenous labor choice: Bodie, Merton, and Samuelson (1992), Basak (1999), Liu and Neis (2002)
- Early retirement: Panageas and Farhi (2003)

None of these papers imposes borrowing constraint against labor income or considers insurance or bequest at a random death time.

Specialized assumptions for most of today's presentation

Today's presentation focus on versions of the model which can be solved more or less completely.

- three cases
  - benchmark: mandatory retirement date and borrowing permitted
  - NBC: No Borrowing Constraint, choice of when to retire
  - BC: Borrowing Constraint, choice of when to retire
- stationary economic environment
  - constant mortality rate, interest rate, and pure rate of time discount
  - stationary joint lognormal salary and stock price
  - insurance is fairly priced and continuously available

Main results

- explicit parametric solutions up to at most one constant
- flexible retirement and borrowing against labor income are valuable and significantly change the optimal consumption and investment policy
- labor income uncertainty may invalidate the traditional life cycle investment advice of reducing stock investment as one gets old
- consumption as a fraction of wealth jumps down (up) at voluntary retirement date for an investor with relative risk aversion greater (less) than 1.
- stock investment as a fraction of wealth jumps down at voluntary retirement date.

Notation

- $u(c_t) = \frac{{c_t}^{1-\gamma}}{1-\gamma}$ – felicity of consumption at time t if not retired
- $u(Kc_t)$ – felicity of consumption at time t if retired, $K > 1$
- $u(kB_t)$ – utility contribution of bequest at t
- ρ – pure rate of time discount
- δ – hazard rate for mortality (constant for this version of the model)
- r – riskfree rate
- $dS_t/S_t = \mu dt + \sigma dZ_t$ – stock price process
- $\kappa \equiv (\mu - r)/\sigma$ – price of risk
- θ_t – holding in the stock (wealth units)
- $dy_t/y_t = \mu_y dt + \sigma_y dZ_t$ – labor income process (constant these slides)
- τ – voluntary retirement date
- T – mandatory retirement date
- W^0 – initial wealth

Choice problems

Choose the retirement date τ (a stopping time), the adapted consumption process c_t, the adapted bequest B_t, and the adapted portfolio choice θ_t to maximize

$$E\left[\int_{t=0}^{\tau} e^{-(\rho+\delta)t}\left(u(c_t)+\delta u(kB_t)\right)dt+\int_{t=\tau}^{\infty} e^{-(\rho+\delta)t}\left(u(Kc_t)+\delta u(kB_t)\right)dt\right]$$

subject to the budget conditions on the wealth process

$$W_0 = W^0$$

$$dW_t = \begin{cases} (rW_t+\delta(W_t-B_t))dt+\theta_t\left((\mu-r)dt+\sigma dZ_t\right)-c_t dt+y dt & t\leq\tau \\ (rW_t+\delta(W_t-B_t))dt+\theta_t\left((\mu-r)dt+\sigma dZ_t\right)-c_t dt & t\geq\tau \end{cases}$$

additional constraints for each of three cases

benchmark: $(\forall t\geq 0)W_t \geq -\frac{y}{r+\delta}$ and $\tau = T$
NBC: $(\forall t\geq 0)W_t \geq -\frac{y}{r+\delta}$
BC: $(\forall t\geq 0)W_t \geq 0$

About the solution

After retirement (all three problems), the agent's problem is a slight variant of the Merton consumption withdrawal model with CRRA, fixed interest rate, and i.i.d. returns. A complete solution is available.

Before retirement, the three cases are different:

- benchmark case: as after retirement, the agent's problem is a slight variant of the Merton model, and a complete solution is available.
- NBC case: a change of variable to the dual state variable equal to the marginal utility of wealth transforms the Bellman equation into a linear equation that can be solved exactly. The two free parameters in the solution of the Bellman equation are determined by (i) smooth pasting at the free boundary for retirement and (ii) a growth condition for small wealth. We have the exact solution in the dual, which is a parametric solution to the primal problem.
- BC case: the same Bellman equation as in the NBC case but two free boundaries: (i) smooth pasting at the free boundary for retirement and (ii) smooth pasting at the free boundary corresponding to zero wealth.

Sample solution: NBC case (Theorem 2)

Start with the dual variable

$$x_t = x_0 \exp\left((r - \rho - \frac{1}{2}\kappa^2)t - \kappa Z_t\right)$$

where x_0 solves $-y_0\phi_x(x_0) = W^0$ for the dual value function ϕ is

$$\phi(x) = \begin{cases} -\hat{\eta}\frac{x^b}{b} & x \le \underline{x} \\ A_+ x^{\alpha_-} - \eta\frac{x^b}{b} + \frac{1}{\beta_2+\beta_3}x & x > \underline{x} \end{cases}$$

with the various constants defined on the following slide. Then the optimal consumption policy is

$$c_t{}^* = \begin{cases} y x_t{}^{-1/\gamma} & t < \tau^* \\ K^{-b} y x_t{}^{-1/\gamma} & t \ge \tau^* \end{cases}$$

with optimal risky portfolio

$$\theta_t^* = y\sigma^{-1}(\mu - r\mathbf{1})x_t\phi_{xx}(x_t),$$

optimal bequest

$$B_t^* = k^{-b} y x_t^{-1/\gamma},$$

optimal retirement policy

$$\tau^* = \inf\{t \ge 0 : x_t \le \underline{x}\},$$

and wealth process

$$W_t^* = -y\phi_x(x_t).$$

In addition the value function is

$$V(W,y) = y^{1-\gamma}(\phi(x) - x\phi_x(x))$$

where x solves $-y\phi_x(x) = W$.

Some constants for the solution

$$b = 1 - 1/\gamma$$

$$\kappa = \sigma^{-1}(\mu - r\mathbf{1})$$

$$\eta = \frac{\gamma(1 + \delta k^{-b})}{\rho + \delta - (1-\gamma)(r + \delta + \frac{\kappa^2}{2\gamma})}$$

$$\hat{\eta} = \frac{\gamma(K^{-b} + \delta k^{-b})}{\rho + \delta - (1-\gamma)(r + \delta + \frac{\kappa^2}{2\gamma})}$$

$$\alpha_{\pm} = \frac{\beta_2 + \frac{1}{2}\beta_1 \pm \sqrt{(\beta_2 + \frac{1}{2}\beta_1)^2 + 2\beta_3\beta_1}}{\beta_1}$$

$$f(t) = \left((\hat{\eta} - \eta) \exp\left(\frac{1 + \delta k^{-b}}{\eta}(T - t) \right) + \eta \right)$$

$$\beta_1 = \kappa^2$$

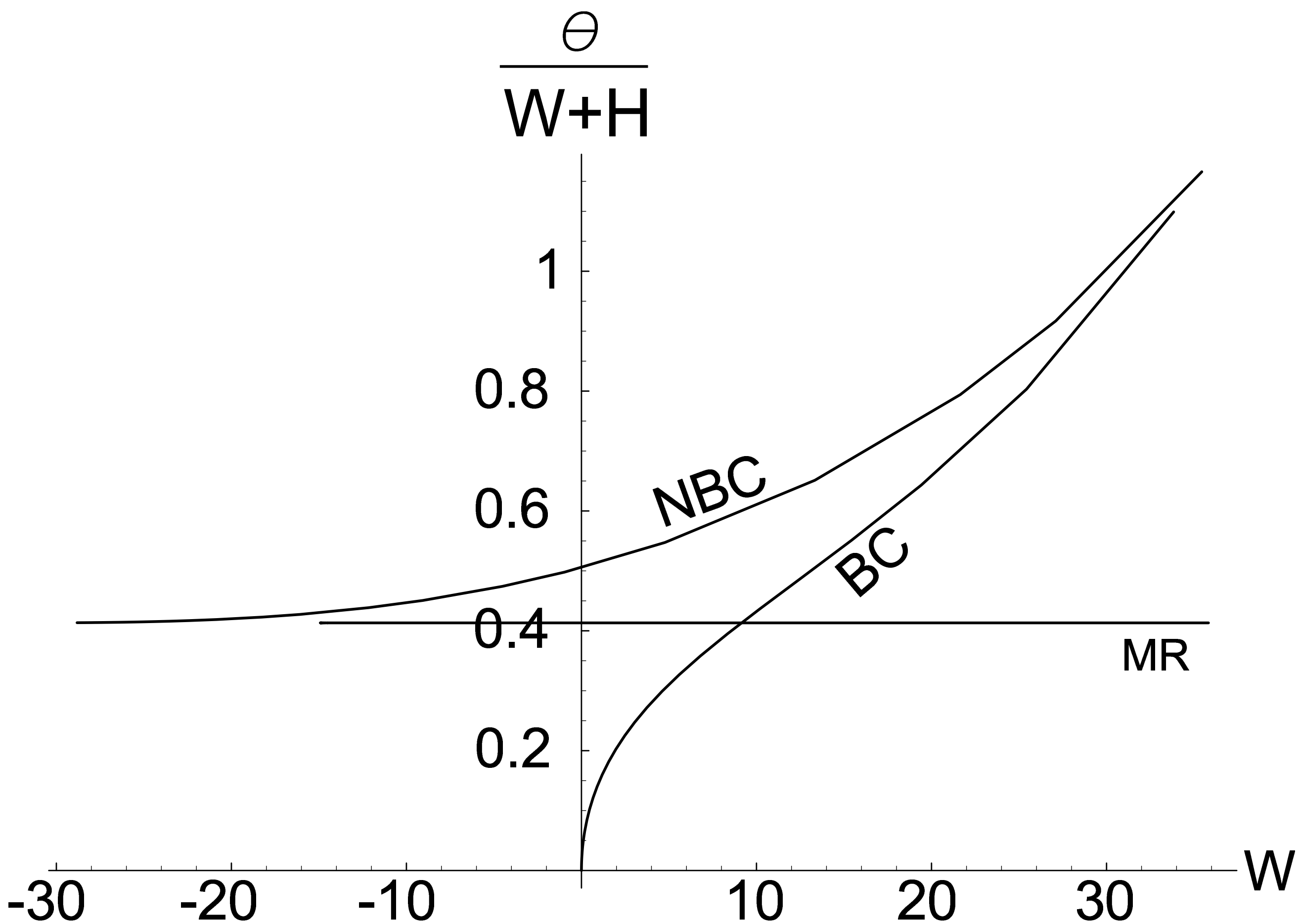
$$\beta_2 = r - \rho$$

$$\underline{x} = \left(\frac{(\eta - \hat{\eta})(b - \alpha_-)(\beta_2 - \beta_3)}{b(1 - \alpha_-)} \right)^{\gamma}$$

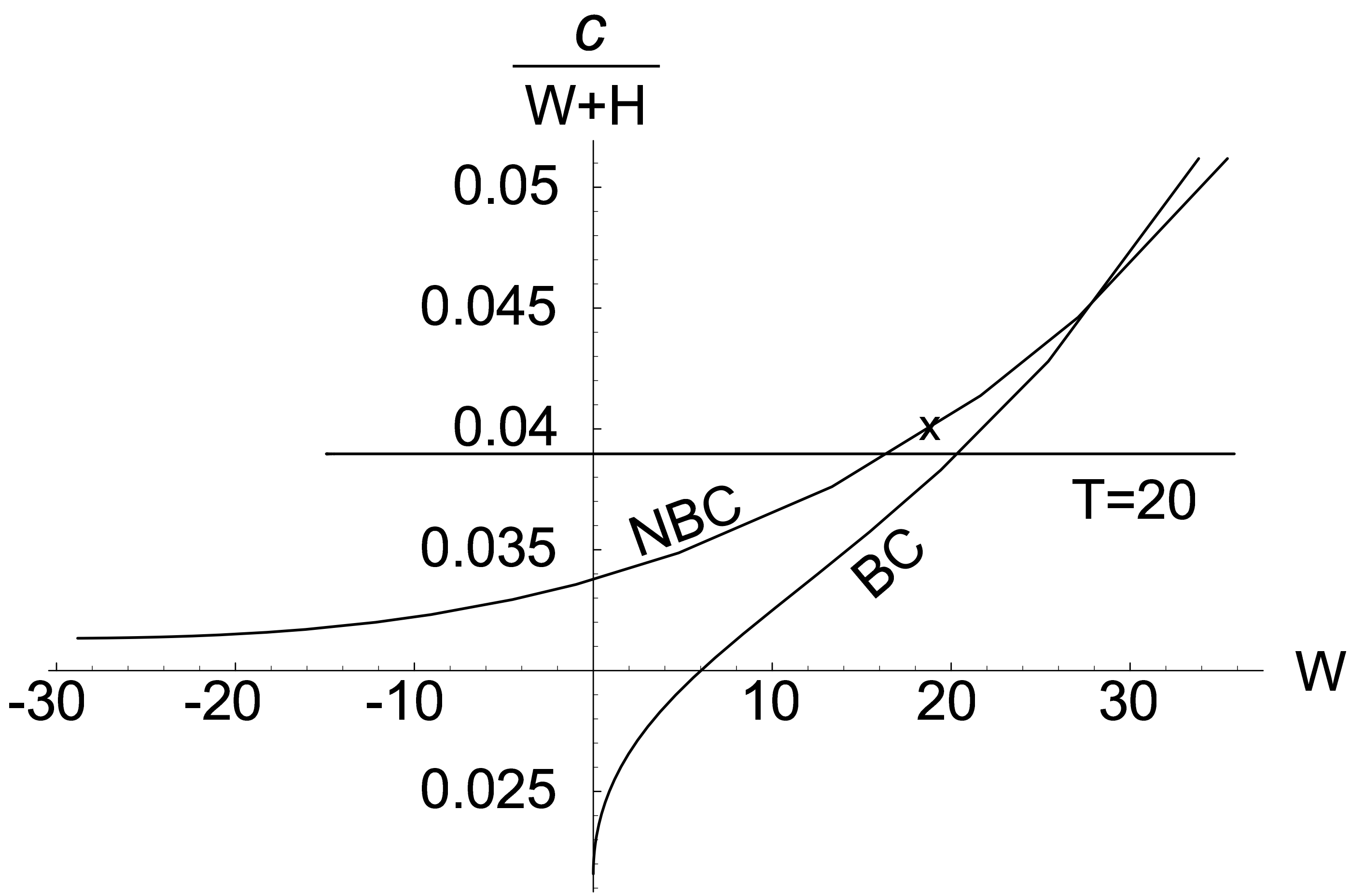
$$A_+ = \frac{\eta - \hat{\eta}}{b}\underline{x}^{b-\alpha_-} - \frac{1}{\beta_2 + \beta_3}\underline{x}^{1-\alpha_-}$$

Default parameters for the plots

- $\gamma = 2$ – coefficient of relative risk aversion
- $K = 3$ – preference for not working
- $k = 5\%$ – bequest scale
- $\rho = 1\%/\text{year}$ – pure rate of time discount
- $\delta = 2\%/\text{year}$ – hazard rate of mortality
- $r = 1\%/\text{year}$ – interest rate
- $\mu = 5\%/\text{year}$ – mean stock return
- $\sigma = 22\%/\sqrt{\text{year}}$ – standard deviation of stock return
- $y_0 = 1/\text{year}$ – constant wage income (equivalently wealth is in units of the multiple of income)

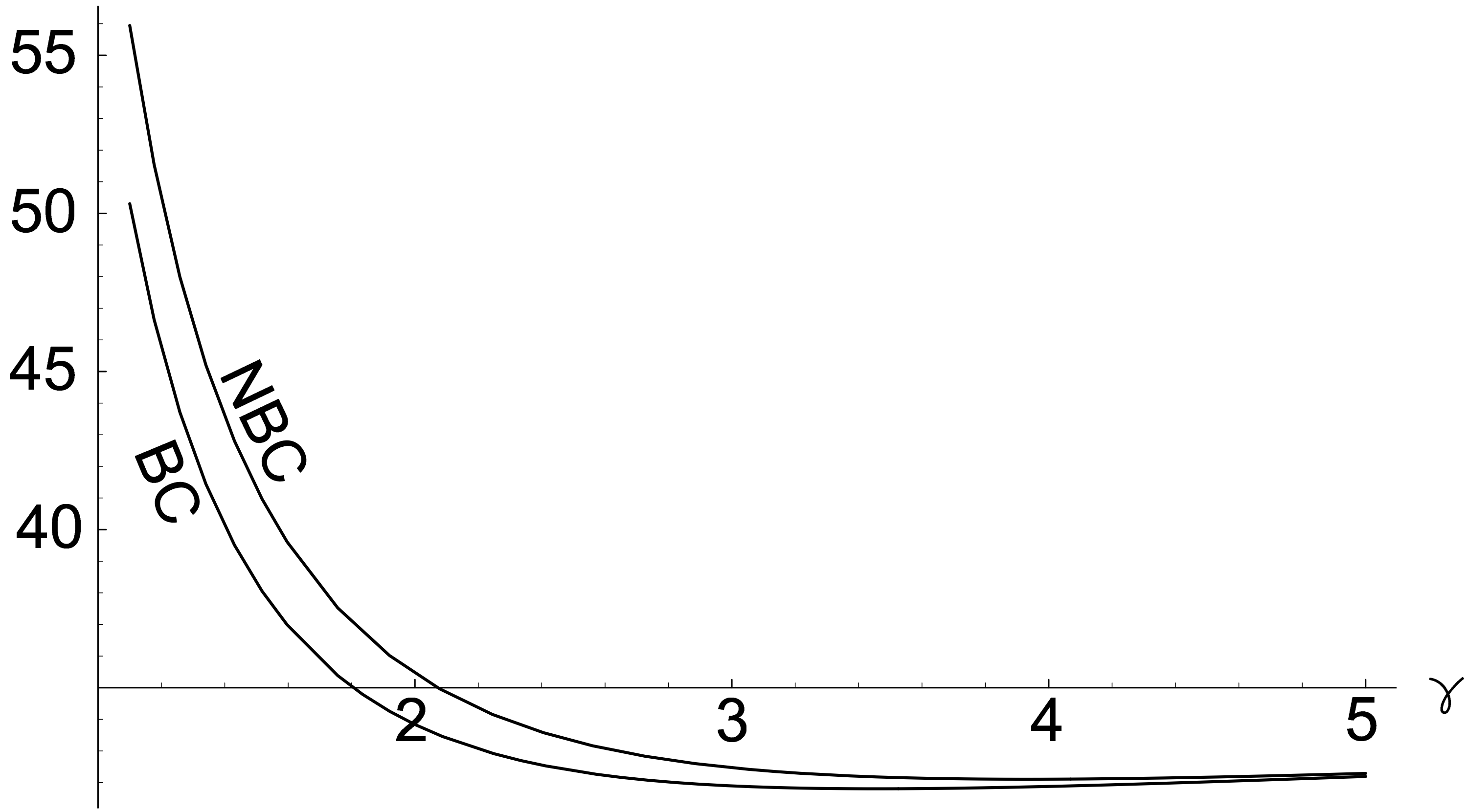


Equity θ per unit of total wealth (financial wealth W + human capital H), as a function of the financial wealth W.

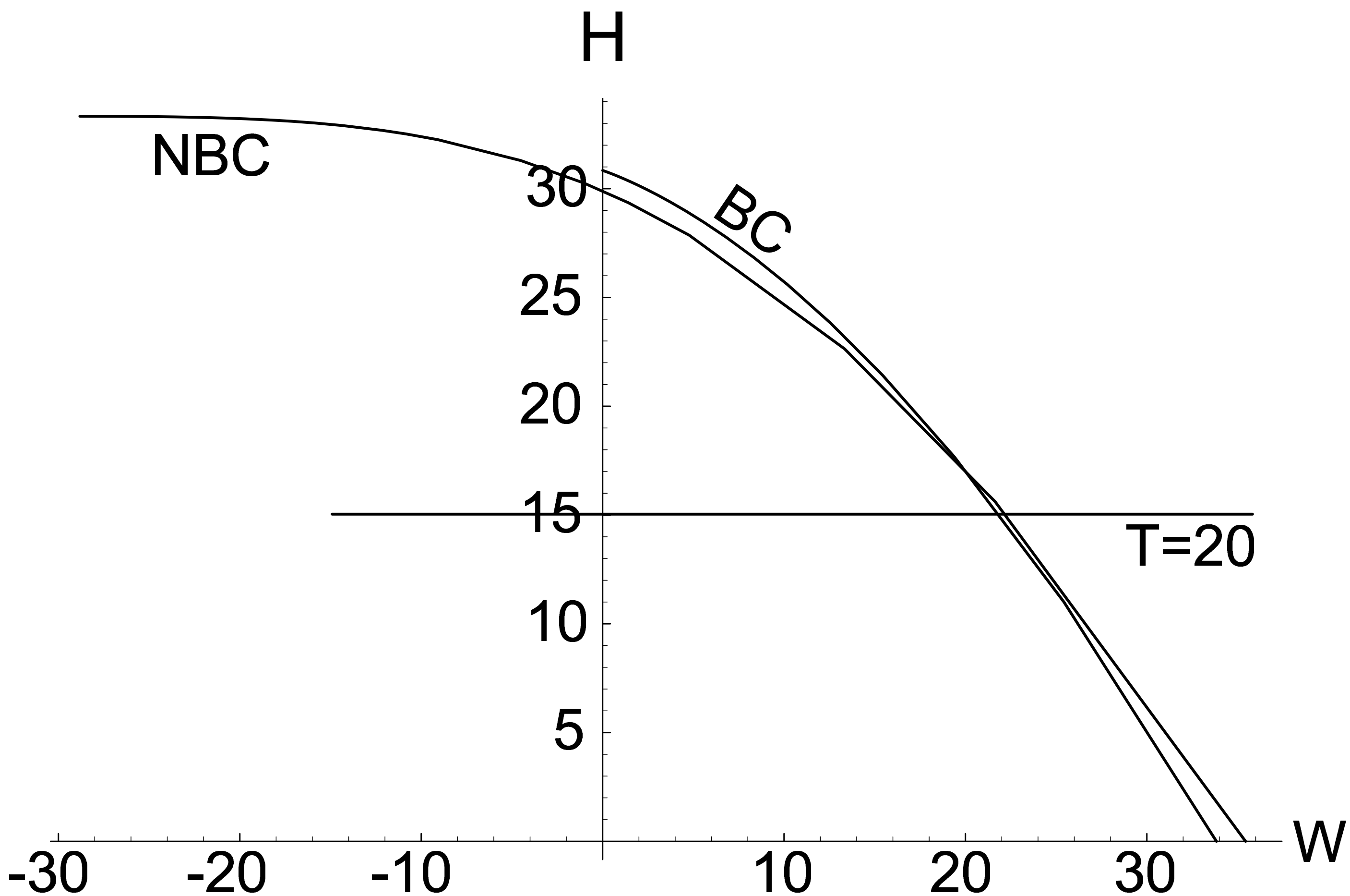


Consumption c rate per unit of total wealth $W+H$, as a function of financial wealth W.

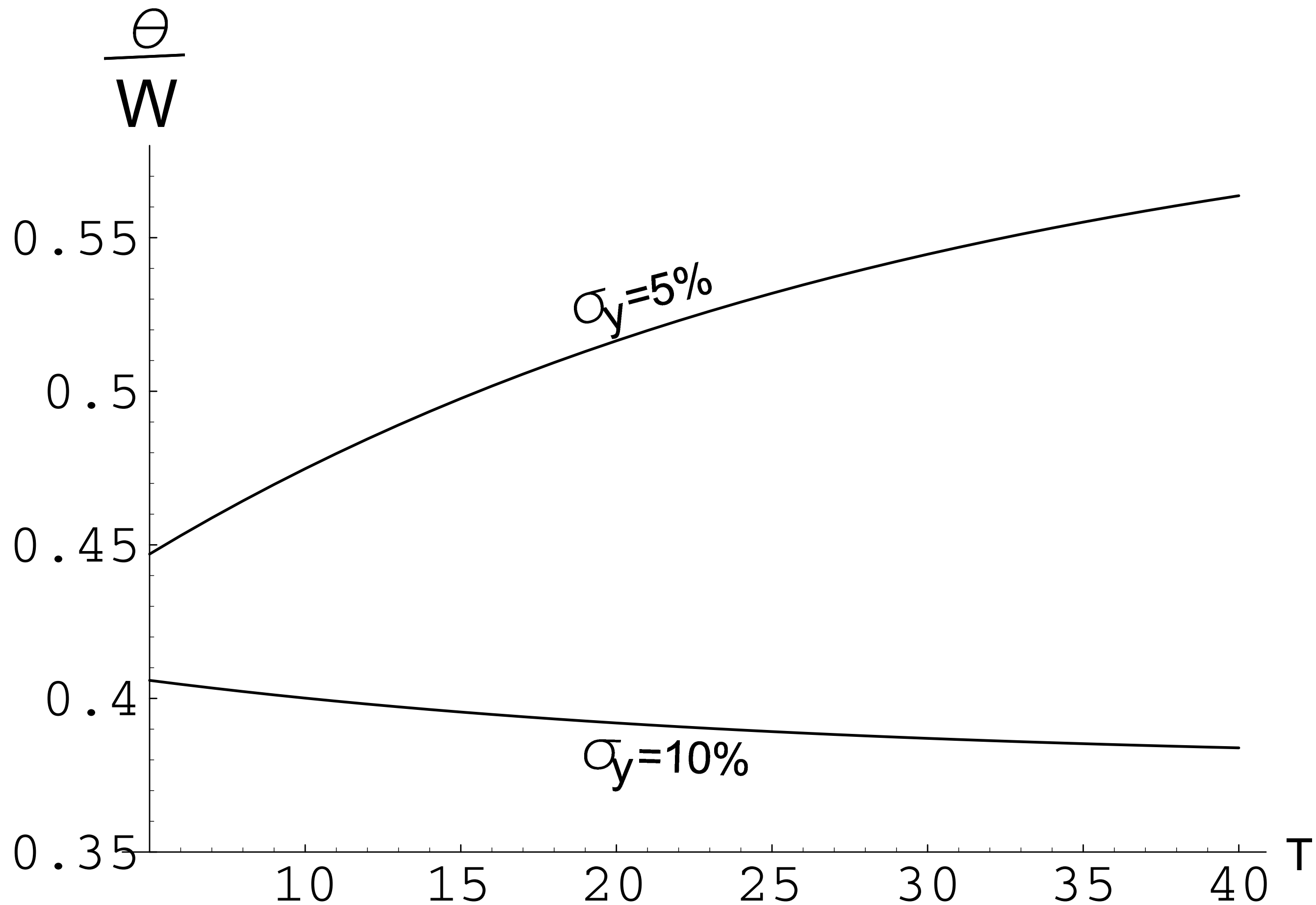
Financial Wealth Threshold for Retirement



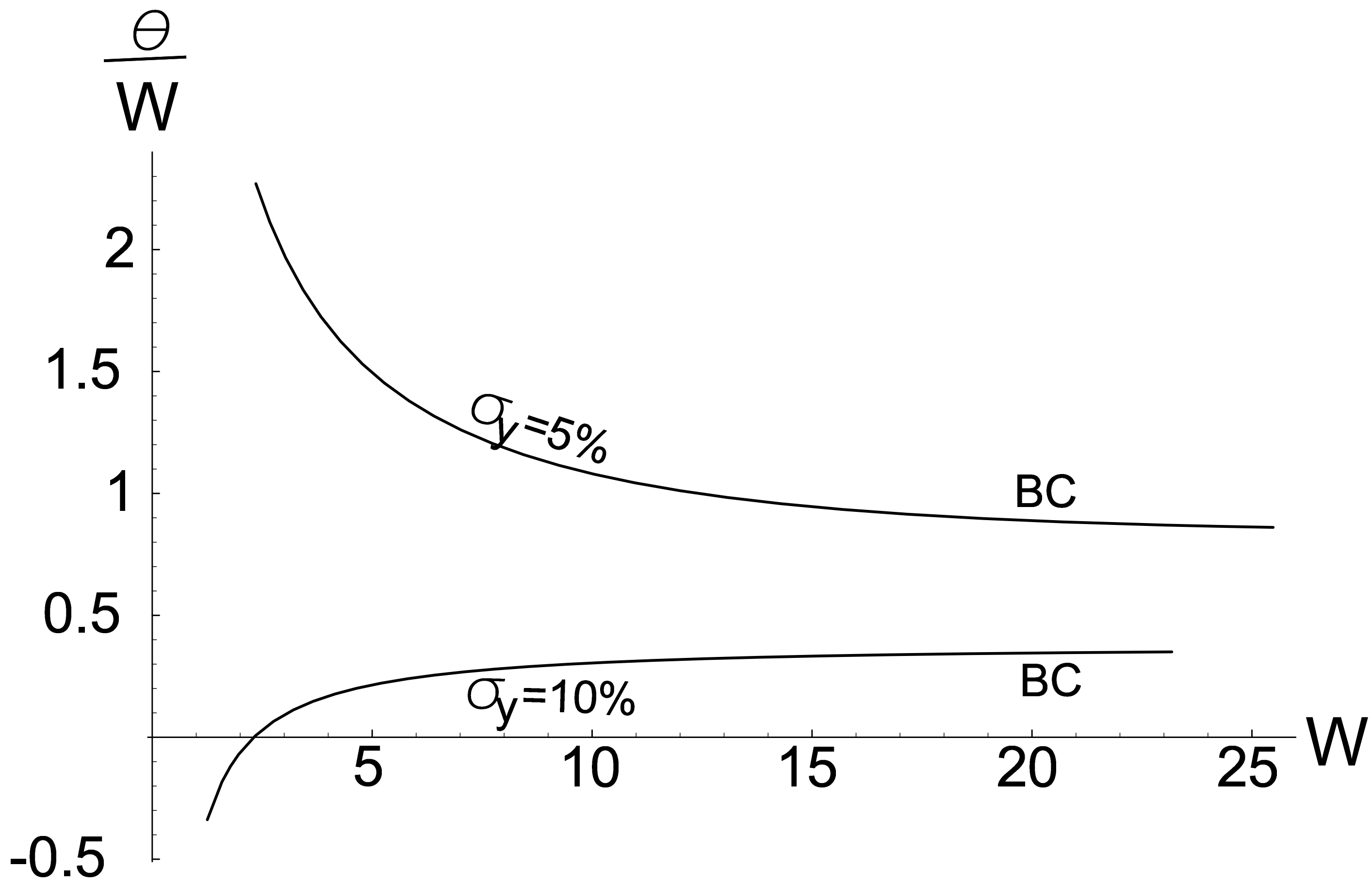
Financial wealth threshold for retirement as a function of relative risk aversion γ for NBC and BC cases with a choice of retirement



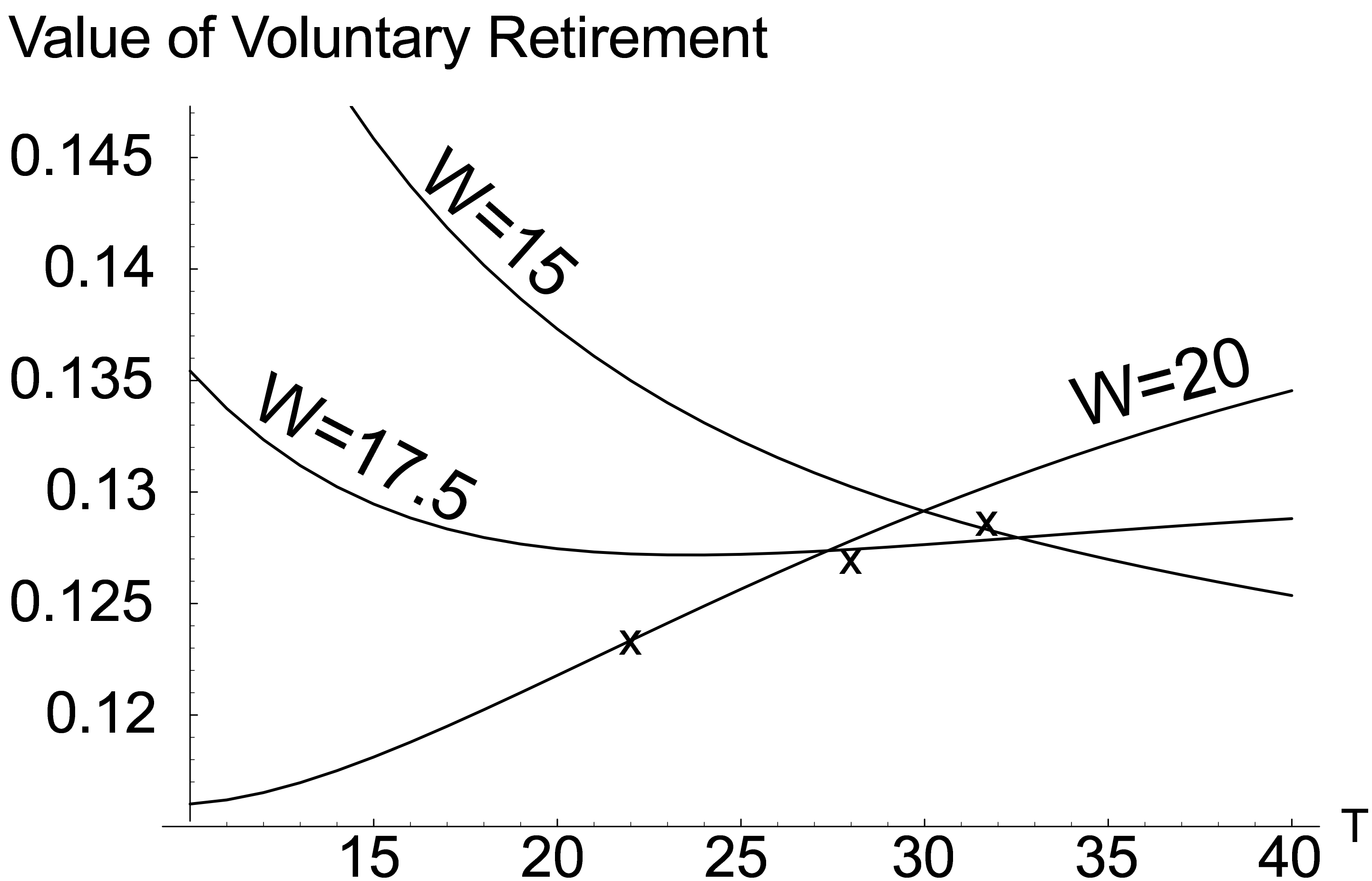
Human capital as a function of financial wealth.



Risky asset holding per unit of financial wealth when the salary has market risk (benchmark case): contradition of the traditional advice that people near retirement should hold less stock.

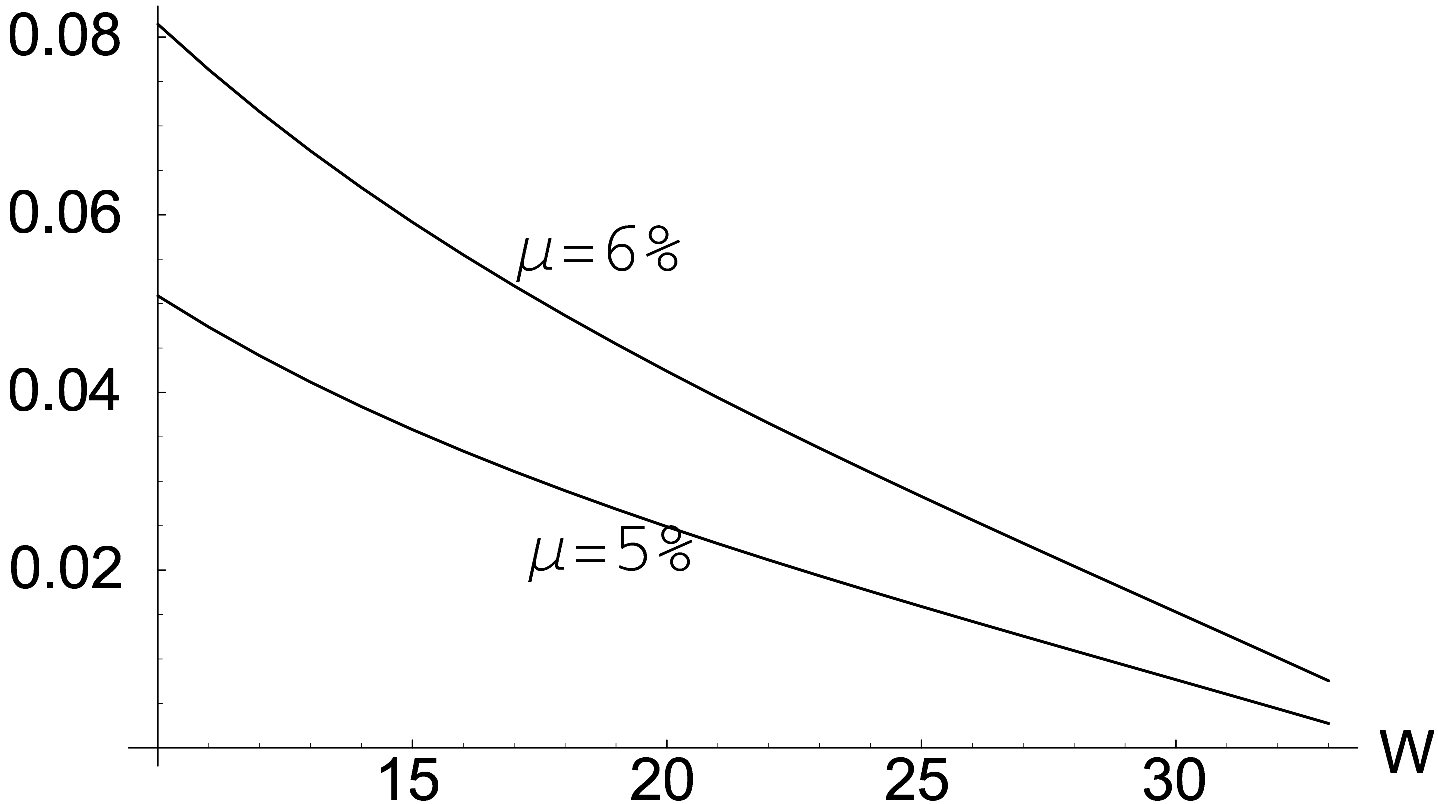


Risky asset holding per unit of financial wealth when the salary has market risk (BC case): contradition of the traditional advice that people near retirement should hold less stock.

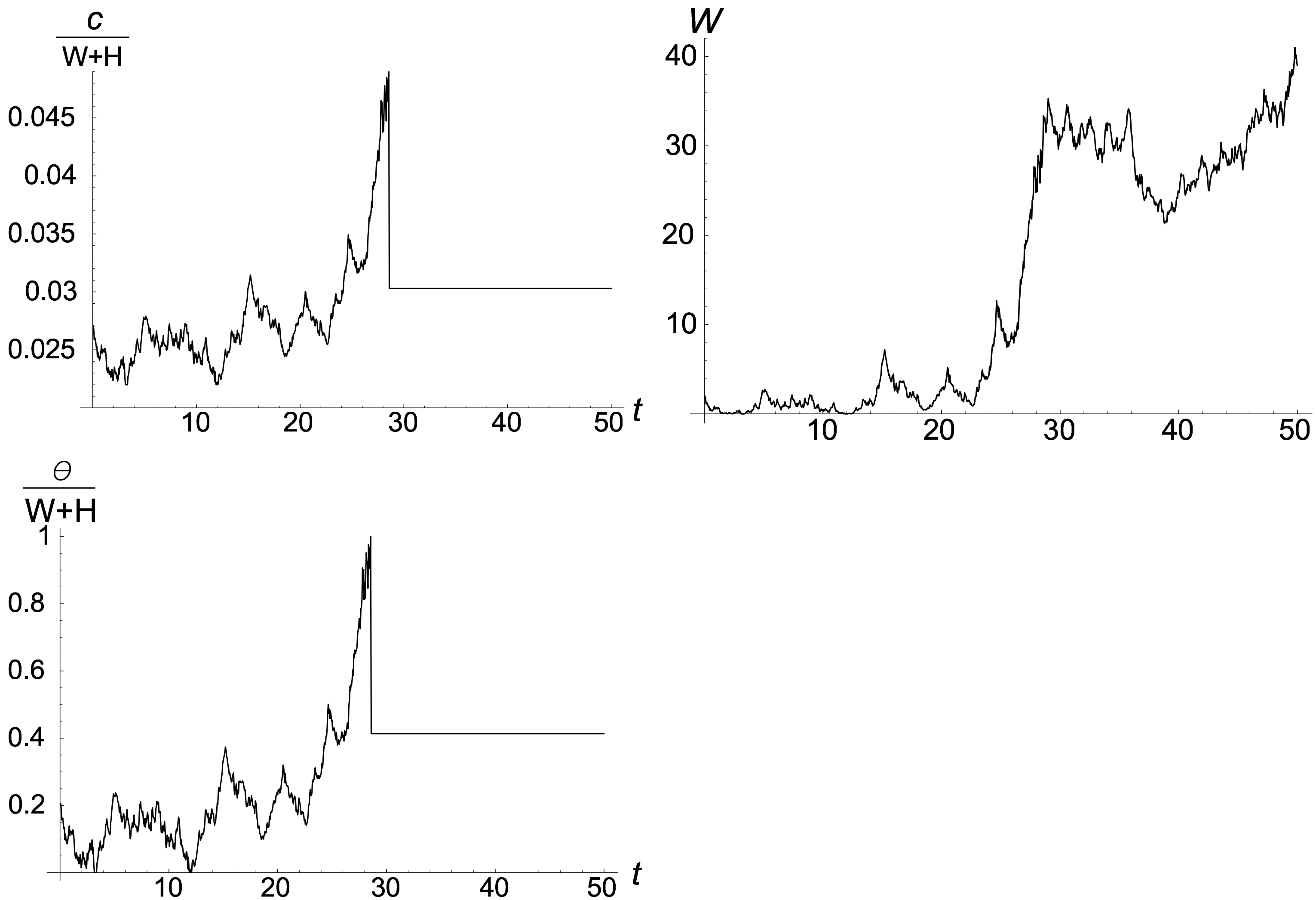


Certainty-equivalent value of voluntary retirement per unit of total wealth, as a function of retirement horizon T.

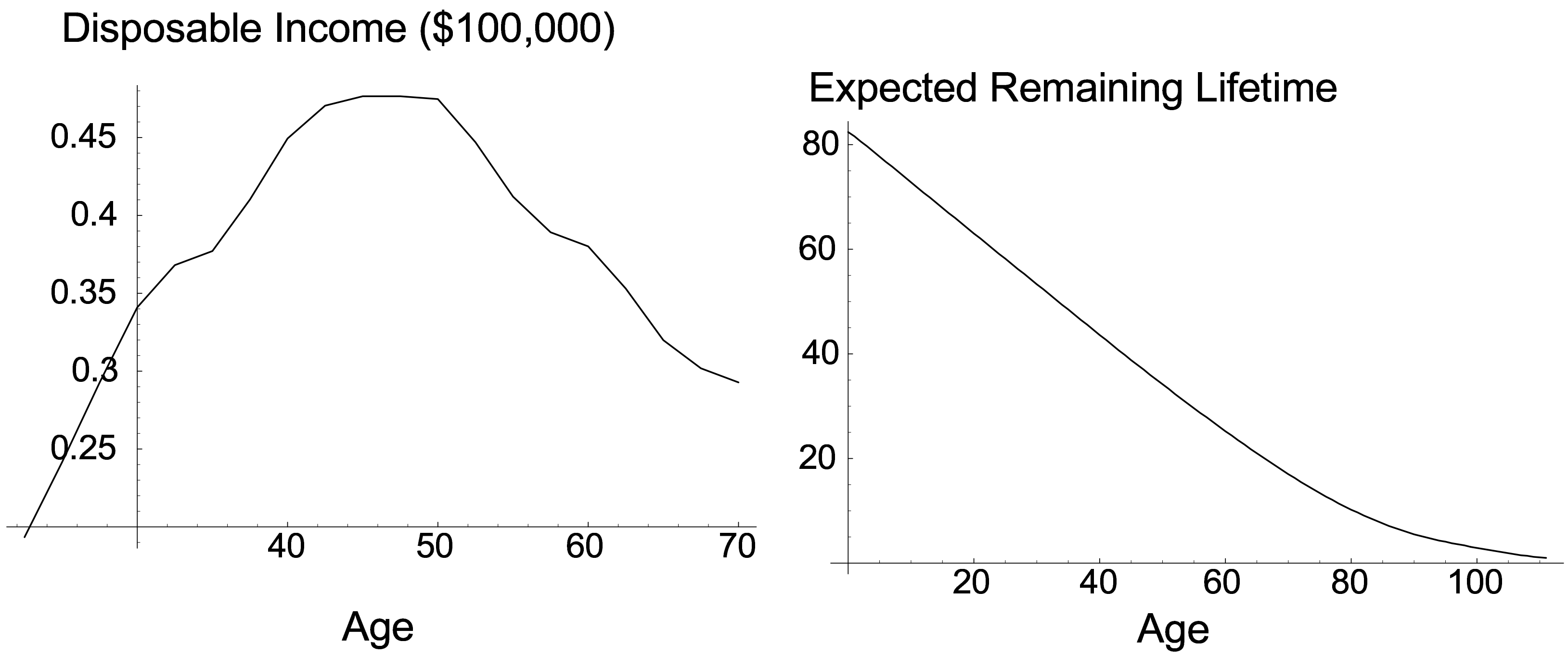
Cost of Borrowing Constraint



Certainty-equivalent value of borrowing per unit of total wealth, as a function of financial wealth W.

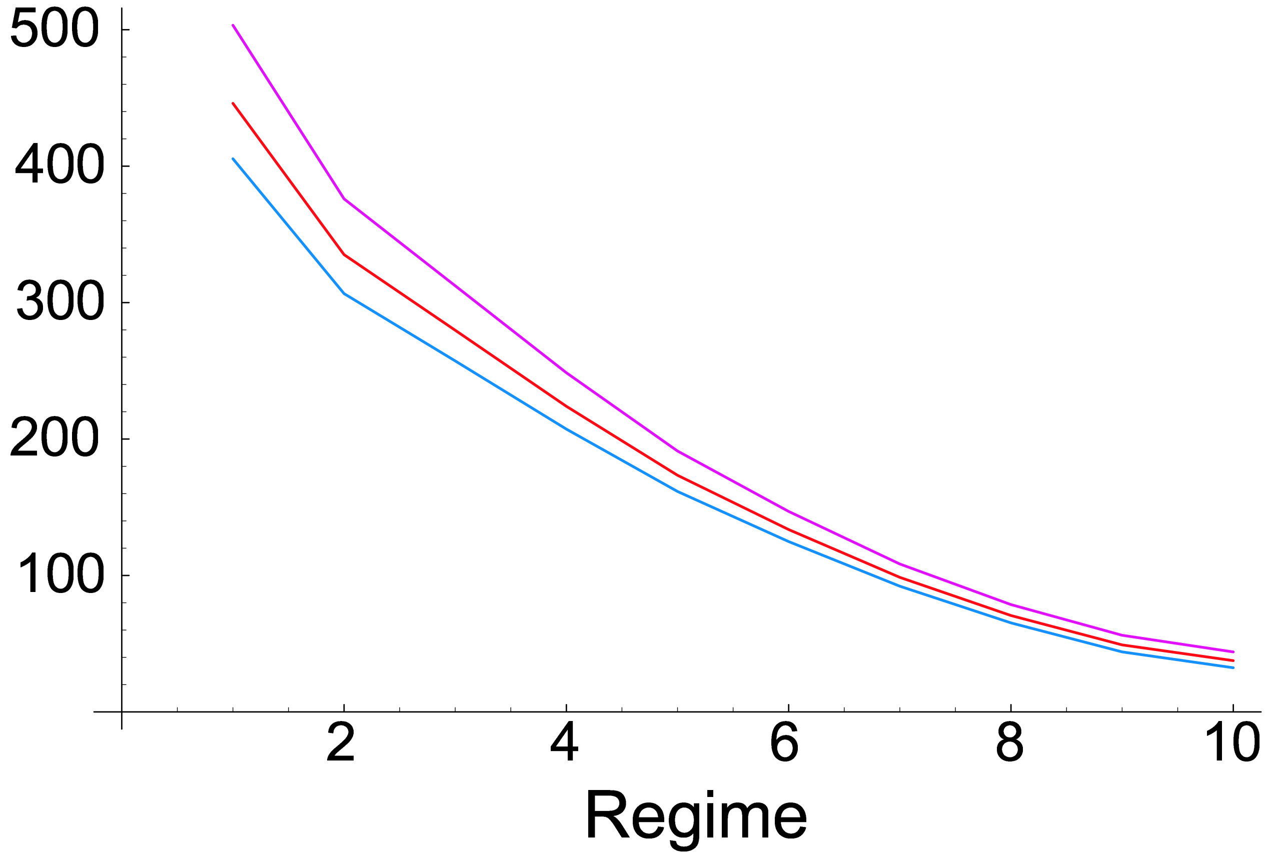


$C/(W+H)$, W, and $\theta/(W+H)$ in one simulated path



Our original analysis assumed a lifetime income and mortality profile similar to empirical data.

Wealth threshold for retirement



Retirement is age dependent in the original analysis. We think of this as an intermediate case between the stationary voluntary retirement case in most of these slides (which has a horizontal boundary) and the mandatory retirement case (which has a vertical boundary).

Conclusion

We have constructed workhorse models of lifetime consumption and investment for analyzing retirement, pensions, and insurance. The versions presented here feature:

- explicit parametric solutions up to at most one constant
- flexible retirement and borrowing against labor income are valuable and significantly change the optimal consumption and investment policy
- labor income uncertainty may invalidate the traditional life cycle investment advice of reducing stock investment as one gets old
- consumption as a fraction of wealth jumps down (up) at voluntary retirement date for an investor with relative risk aversion greater (less) than 1.
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