Transfer of non-tradeable risk

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Agenda

⋆ Main motivation of this study and some references.

⋆ **Static Framework :**

- ⋄ Risk measures and indifference pricing.
- ⋄ Risk measures and hedging.
- ⋄ Optimal derivatives design as an inf-convolution problem.

⋆ **Dynamic framework :**

- ⋄ Axiomatic approach of dynamic convex risk measures.
- ⋄ Dynamic convex risk measures and BSDEs : g-conditional risk measures.
- ⋄ Inf-convolution of g-conditional risk measures.

Introduction and motivation

Development of new financial products

A new type of assets :

– Recent introduction of a new type of financial contracts with a non-financial underlying risk : ”cat-bonds”, weather derivatives...
– *Illiquid* instruments, with an underlying asset which is *not traded* on financial markets.
– Underlying risk possibly related to a financial market risk.

Convergence and interplay between finance and insurance :

⇒ Use of the knowledge of financial risk management to the management of other kinds of risk.

⇒ Use of the insurance technology to design structured products.

Main questions

– From a financial point of view :

  - What is the *pricing rule* ?
  - What is the *hedging* strategy ?

– From an insurance point of view :

  What is the *"optimal" structure to issue or "optimal" transfer of a non-tradable risk* so that a transaction occurs ?

⇒ The notion of "optimality" requires a *choice criterion* : *convex risk measures.*

⇒ The point of view adopted here is *global* : analysis of the *global* transaction.

Some related works (among many others...!!)

⋆ *Insurance literature on optimal policy design* : Borch (1962), Raviv (1979), Gerber (1980), Bühlmann (1970)...

⋆ *Indifference pricing* (often in an exponential framework) :
Hodges-Neuberger(1989), Davis(1997), Rouge-El Karoui(2000), Becherer (2001), Delbaen et al (2002), Musiela-Zariphopoulou (2004)....

⋆ *Risk measures* (seminal papers) : Artzner et al.(1999), Föllmer-Schied (2002), Frittelli-Gianin (2002)...
(multi-period setting) : Artzner et al.(2004), Cheridito-Delbaen-Kupper (2004), Riedel (2004), Weber (2004)...

⋆ *BSDEs and finance* : El Karoui-Quenez (1996), El Karoui-Peng-Quenez (1997), Peng (1997, 2003), Mania-Schweizer (2004)...

Static Framework

Indifference Pricing and Convex Risk measures

Indifference Price

⋆ Let us consider an agent with a risky position W and a utility function u.

⋆ **Definition** : the buyer's indifference price of a claim X corresponds to the maximal amount π, the agent is ready to pay for X :

$$\boxed{\mathbb{E}[u(W + X - \pi)] = \mathbb{E}[u(W)].}$$

This is not a transaction price. It gives an upper bound to the price of this claim so that a transaction will take place given the agent's initial exposure.

⋆ **Properties** : this price should be :

(a) increasing monotonic,

(b) convex.

(c) *Sometimes*, it is also translation invariant : $\pi(X + m) = \pi(X) + m$.

$\Rightarrow$ In this last case, the indifference price can be seen as a convex premium principle as introduced by Deprez-Gerber (1985).

Example of the exponential utility function :

When $u(x) = -\gamma \exp\Big(-\frac{1}{\gamma}x\Big)$ with γ the *risk-tolerance* coefficient,

$$\pi(X) = e_\gamma\Big(W\Big) - e_\gamma\Big(W + X\Big),$$

where e_γ is the opposite of the certainty equivalent, defined as follows :

$$e_\gamma\Big(\Psi\Big) \equiv \gamma \ln \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-\frac{1}{\gamma}\Psi\Big)\Big].$$

Risk measures : an introduction

⋆ The most famous risk measure is the so-called **V@R** (Value at Risk) defined as the smallest amount to be added to a certain position X to make it acceptable for a given level ε :

$$\boxed{V@R_{\varepsilon}(X) = \inf\{k : \mathbb{P}(X + k < 0) \leq \varepsilon\}.}$$

The V@R has several *key properties* :

- ⋄ *decreasing monotonicity*,
- ⋄ *translation invariant* : $\forall m \in \mathbb{R} \quad V@R_{\varepsilon}(X + m) = V@R_{\varepsilon}(X) - m$,
- ⋄ *positive homogeneity* : $\forall \lambda \geq 0 \quad V@R_{\varepsilon}(\lambda X) = \lambda V@R_{\varepsilon}(X)$.

⇒ It is not however **convex**.

The convexity property is crucial to take into account the impact of diversification on risk.

⇒ The idea is then to generalize the notion of risk measures to convex functionals, being decreasing and translation invariant (Artzner *et al.* (1999), Föllmer and Schied (2002), Frittelli and Gianin (2002)...).

Risk measures as an indifference price

Let $(\Omega, \mathcal{F})$ be a standard measurable space and $\mathcal{X}$ the linear space of bounded functions (including constant functions).

Definition : *The functional ρ is a* <u>*convex risk measure*</u> *if it satisfies the following properties :*

(a) *Convexity ;*

(b) *Decreasing monotonicity ;*

(c) *Translation invariance :* $\forall X \in \mathcal{X}, \forall m \in \mathbb{R}, \quad \rho(X+m) = \rho(X) - m.$

Using (c), $\boxed{\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.}$

$\Rightarrow -\rho(X)$ can be seen as the maximum amount the agent is ready to pay to hold the risk X in her portfolio.

$\Rightarrow -\rho(X)$ can be seen as the *indifference buyer's price*.

Dual representation

The convexity of the framework leads to an "explicit" representation of the convex risk measures :

Theorem : *There exists a* *penalty function* α *taking values in* $\mathbb{R} \cup \{+\infty\}$ *such that :*

$$\boxed{\begin{array}{l} \forall \Psi \in \mathcal{X}, \quad \rho(\Psi) = \sup_{\mathbf{Q} \in \mathcal{M}_{1,f}} \{\mathbb{E}_{\mathbf{Q}}[-\Psi] - \alpha(\mathbf{Q})\}, \\ \forall \mathbf{Q} \in \mathcal{M}_{1,f}, \quad \alpha(\mathbf{Q}) = \sup_{\Psi \in \mathcal{X}} \{\mathbb{E}_{\mathbf{Q}}[-\Psi] - \rho(\Psi)\} \end{array}}$$

where $\mathcal{M}_{1,f}$ *is the set of all additive measures on* $(\Omega, \mathcal{F})$.
Moreover, the supremum is attained in $\mathcal{M}_{1,f}$ *and*

$$\forall \Psi \in \mathcal{X} \quad \rho(\Psi) = \max_{\mathbf{Q} \in \mathcal{M}_{1,f}} \Big\{ \mathbb{E}_{\mathbf{Q}}[-\Psi] - \alpha\Big(\mathbf{Q}\Big) \Big\}.$$

Example : the entropic risk measure

The entropic risk measure is defined as $\forall \Psi \in \mathcal{X}, \quad e_\gamma(\Psi) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\gamma}\Psi\right)\right]$.

Its dual representation is :

$$\boxed{e_\gamma(\Psi) = \sup_{\mathbb{Q} \in \mathcal{M}_1} \left(\mathbb{E}_{\mathbb{Q}}[-\Psi] - \gamma h(\mathbb{Q}|\mathbb{P})\right)}$$

where $\mathcal{M}_1$ is the set of all probability measures on $(\Omega, \mathcal{F})$ and $h(\mathbb{Q}|\mathbb{P})$ is the relative entropy of $\mathbb{Q}$ w.r. to $\mathbb{P}$.

Risk measures and hedging

$$\boxed{\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.}$$

$\Rightarrow \rho(X)$ can be also interpreted as the *minimal capital requirement* to be added to the position X to make it "acceptable" : the new position $X + \rho(X)$ does not carry any positive risk.

⋆ The *acceptance set* associated with ρ is defined as : $\mathcal{A}_\rho = \{\Psi \in \mathcal{X}, \quad \rho(\Psi) \leq 0\}$.

⋆ Another characterization of the risk measure ρ is then :

$$\rho(X) = \inf\{m \in \mathbb{R}, m + X \in \mathcal{A}_\rho\}.$$

Super-hedging as risk measures

⋆ Let $\mathcal{H}$ be a convex subset of $\mathcal{X}$ consisting of potential hedges. Then,

$$\{\Psi \in \mathcal{X}, \exists H \in \mathcal{H}, \quad \Psi \geq H\} \text{ is the } \textit{set of all super-hedged positions.}$$

⇒ It is possible to introduce the convex risk measure $\nu^{\mathcal{H}}$ defined by :

$$\boxed{\begin{aligned} \nu^{\mathcal{H}}(\Psi) &= \inf\{m \in \mathbb{R}; \text{such that } \exists H \in \mathcal{H},\ m + \Psi \geq H\} \\ &= \inf_{H \in \mathcal{H}} \sup_{\omega}(-\Psi + H)(\omega) = \inf_{H \in \mathcal{H}} \rho_{\max}(\Psi - H), \end{aligned}}$$

where $\rho_{\max}(\Phi) = \sup_{\omega \in \Omega}\{-\Phi(\omega)\}$ is the *worst case risk measure.*
⇒ The associated penalty function is $\alpha^{\mathcal{H}}(\mathbf{Q}) = \sup_{H \in \mathcal{H}} \mathbf{E}_{\mathbf{Q}}[-H]$.

⋆ **Interpretation :**
⇒ $\nu^{\mathcal{H}}$ can be interpreted as the worst case risk measure $\rho_{\max}$ reduced by the use of hedging variables in $\mathcal{H}$.
⇒ $-\nu^{\mathcal{H}}(\Psi)$ can be seen as the *super buyer's price* (or *lower hedging price*) of Ψ :

$$\pi_b^{\mathcal{H}}(\Psi) = \inf\{y \in \mathbb{R}; \text{such that } \exists H \in \mathcal{H},\ \Psi + y - H \geq 0\}$$

(smallest initial value of the super-hedging portfolio for the claim Ψ).

Optimal Derivative Design

General framework

⋆ **Agent A** : at a future time T, agent A is exposed towards a non-tradable risk for an amount $\mathbf{X_T^A}$. She calls on an investor, agent B, to reduce her exposure by the *sale of a structured contract* **F**.

⋆ **Agent B** : She has her own exposure $\mathbf{X_T^B}$ at time T. She pays a (forward) premium $\boldsymbol{\pi}$ at time 0 and receives in exchange the structure F at time T.

⋆ Both agents assess their risk using convex risk measures, $\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_B}$.

⇒ The problem is to find the optimal structure (F, π).

Transaction feasibility

Relationship between both agents : *Both agents do not have the same goals.*

⋆ Agent A looks for a hedge of her exposure. Equivalently, she wants to determine the structure (F, π) as to minimize her risk measure :

$$\inf_{F \in \mathcal{X}, \pi} \rho_A(X_T^A - F + \pi).$$

⋆ Agent B wants to improve her risk measure by doing the F-transaction. Her interest in doing the transaction may be written as :

$$\rho_B(X_T^B + F - \pi) \leq \rho_B(X_T^B).$$

Optimal pricing rule : Binding this constraint leads immediately to the optimal pricing rule :

$$\boxed{\pi_B(F) = \rho_B\Big(X_T^B\Big) - \rho_B\Big(X_T^B + F\Big).}$$

$\pi_B(F)$ gives an upper bound to the price agent B is ready to pay to enter the transaction.

Rewording of the optimization program

$$\boxed{\inf_{F\in\mathcal{X},\pi} \rho_A(X_T^A - F + \pi) \quad \text{s.t.} \quad \rho_B(X_T^B + F - \pi) \leq \rho_B(X_T^B).}$$

Using the optimal pricing rule together with the cash translation invariance property :

$$\inf_{F\in\mathcal{X}} \left\{\rho_A\Big(X_T^A - F\Big) + \rho_B\Big(X_T^B + F\Big) - \rho_B\Big(X_T^B\Big)\right\},$$

or to within the constant $\rho_B(X_T^B)$ as :

$$R_{A,B}(X_T^A, X_T^B) := \inf_{F\in\mathcal{X}} \left\{\rho_A\Big(X_T^A - F\Big) + \rho_B\Big(X_T^B + F\Big)\right\}.$$

Equivalently denoting by $\widetilde{F} \equiv X_T^B + F \in \mathcal{X}$:

$$\boxed{R_{A,B}(X_T^A, X_T^B) = \inf_{\widetilde{F}\in\mathcal{X}} \left\{\rho_A(X_T^A + X_T^B - \widetilde{F}) + \rho_B(\widetilde{F})\right\} = \rho_A \Box \rho_B(X_T^A + X_T^B).}$$

$\Rightarrow$ This program is the functional extension of the classical inf-convolution operator acting on real convex functions $f\Box g(x) = \inf_y\{f(x-y) + g(y)\}$.

Optimal risk transfer = Inf-convolution problem.

Interpretation in terms of indifference prices

The optimization program $\inf_{F\in\mathcal{X}}\left\{\rho_A\left(X_T^A - F\right) + \rho_B\left(X_T^B + F\right)\right\}$ is equivalent to :

$$\inf_{F\in\mathcal{X}}\left\{\underbrace{\rho_A\left(X_T^A - F\right) - \rho_A\left(X_T^A\right)}_{=\pi_A^s(F|X_T^A)} + \underbrace{\rho_B\left(X_T^B + F\right) - \rho_B\left(X_T^B\right)}_{=-\pi_B^b(F|X_T^B)}\right\},$$

where :

$\pi_A^s(F|X_T^A)$ is the seller's indifference price of F and,

$\pi_B^b(F|X_T^B)$ is the buyer's indifference price of F.

Equivalently :

$$\sup_{F\in\mathcal{X}}\left\{\pi_B^b\left(F|X_T^B\right) - \pi_A^s\left(F|X_T^A\right)\right\} \quad \geq 0 \quad \left(\text{as for } F\equiv 0, \text{ spread} = 0.\right)$$

$\Rightarrow$ *The transaction may occur when the minimal seller price is less than the maximal buyer price.*

Main tool : Inf-convolution of risk measures

Theorem : *Let ρ_A and ρ_B be two convex risk measures with respective penalty functions α_A and α_B. Let $\rho_{A,B}$ be the inf-convolution of ρ_A and ρ_B defined as*

$$\Psi \to \rho_{A,B}(\Psi) \equiv \rho_A \Box \rho_B(\Psi) = \inf_{H \in \mathcal{X}} \left\{ \rho_A(\Psi - H) + \rho_B(H) \right\}$$

and assume that $\rho_{A,B}(0) > -\infty$.
Then $\rho_{A,B}$ is a convex risk measure which is finite for all $\Psi \in \mathcal{X}$.
The associated penalty function is given by

$$\forall \mathbf{Q} \in \mathcal{M}_{1,f} \quad \alpha_{A,B}(\mathbf{Q}) = \alpha_A(\mathbf{Q}) + \alpha_B(\mathbf{Q}).$$

Corollary : *Let $\nu^{\mathcal{H}}$ be the risk measure generated by a convex subset $\mathcal{H}$ of $\mathcal{X}$, then*

$$\rho \Box \nu^{\mathcal{H}}(\Psi) = \inf \left\{ \rho(\Psi - H), H \in \mathcal{H} \right\}$$

is a convex risk measure with penalty function

$$\forall \mathbf{Q} \in \mathcal{M}_{1,f} \quad \alpha(\mathbf{Q}) + \alpha^{\mathcal{H}}(\mathbf{Q}).$$

Application to optimal hedging : Market modified risk measures

⋆ A typical application of the previous Corollary is the problem of optimal hedging : If the convex set $\mathcal{H}$ represents the set of the gain processes associated with financial investments, then

$$\boxed{\rho\Box\nu^{\mathcal{H}}(\Psi) = \inf\left\{\rho\left(\Psi - \xi\right), \xi \in \mathcal{H}\right\} \overset{def}{=} \rho^{m}\left(\Psi\right).}$$

is a convex risk measure.

⋆ This is the risk measure the agent has after having optimally chosen her financial investment/hedge on the market, simply called market modified risk measure.

Hedging = Inf-convolution problem.

⇒ The financial market plays there the same role as an intermediate agent having a risk measure $\nu^{\mathcal{H}}$.

Risk tolerance coefficient and dilated risk measures

Definition : *Let ρ be a convex risk measure with penalty function α and $\gamma > 0$ a real parameter. γ is the risk tolerance coefficient of any agent having a risk measure ρ_γ associated with the penalty function* $\boxed{\alpha_{\rho_\gamma} = \gamma\ \alpha.}$

Then

$$\boxed{\rho(\Psi) = \sup_{\mathbf{Q}\in\mathcal{M}_{1,f}} \{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \alpha(\mathbf{Q})\} \quad \rightsquigarrow \quad \rho_\gamma(\Psi) = \sup_{\mathbf{Q}\in\mathcal{M}_{1,f}} \{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \gamma\alpha(\mathbf{Q})\}.}$$

⇒ A direct consequence is that the risk measure ρ_γ may be expressed as

$$\boxed{\rho_\gamma\,(\Psi) = \gamma\rho\left(\frac{1}{\gamma}\Psi\right).}$$

It satisfies a *dilatation property* with respect to the size of the position.

⇒ ρ_γ is called the dilated risk measure or the γ-tolerant risk measure associated with ρ.

⇒ Agents having risk measures of the type ρ_γ assess their respective risks using the same family of risk measures but with different coefficients, translating their respective risk tolerance.

Dilated risk measures and inf-convolution

Proposition : *Let* $\left(\rho_\gamma, \gamma > 0\right)$ *be the family of dilated risk measures issued from* ρ. *Then, the following properties hold :*

(i) *For any* $\gamma, \gamma' > 0$,

$$\rho_\gamma \Box \rho_{\gamma'} = \rho_{\gamma+\gamma'}.$$

(ii) *Moreover,*

$$\begin{aligned} \rho_\gamma \Box \rho_{\gamma'}(X) &= \inf_F \left\{\rho_\gamma (X - F) + \rho_{\gamma'}(F)\right\} \\ &= \rho_{\gamma'}(X - F^*) + \rho_{\gamma'}(F^*) \quad \textit{for} \quad F^* = \tfrac{\gamma'}{\gamma+\gamma'} X. \end{aligned}$$

(iii) ρ *is a coherent risk measure if and only if* $\rho_\gamma \equiv \rho$.

(iv) Assume $\rho(0) = 0$. Then ρ_γ is a decreasing function of γ, with asymptotic behavior :

◇ *The marginal risk measure, $\rho_\infty \equiv \lim_{\gamma \to \infty} \rho_\gamma$, is a coherent risk measure and*

$$\boxed{\rho_\infty(\Psi) = \sup_{\mathbf{Q} \in \mathcal{M}_{1,f}} \left\{ \mathbb{E}_{\mathbf{Q}}[-\Psi] \middle| \alpha\Big(\mathbf{Q}\Big) = 0 \right\}.}$$

$\Rightarrow$ *In some cases, and in particular when the the set $\{\mathbf{Q} | \alpha(\mathbf{Q}) = 0\}$ has a single element, $-\rho_\infty$ is a linear pricing rule and can be seen as an extension of the Davis price (In particular : $e_\infty(\Psi) = \mathbb{E}_{\mathbb{P}}(-\Psi)$.)*

◇ *The conservative risk measure ρ_{0^+} defined as the non-decreasing limit of ρ_γ when $\gamma \to 0$ is also a coherent risk measure, that is simply the* "super-pricing rule" *of $-\Psi$:*

$$\boxed{\rho_{0^+}(\Psi) = \lim \uparrow_{\gamma \downarrow 0} \rho_\gamma(\Psi) = \sup_{\mathbf{Q} \in \mathcal{M}_{1,f}} \left\{ \mathbb{E}_{\mathbf{Q}}[-\Psi] \middle| \alpha\Big(\mathbf{Q}\Big) < \infty \right\}.}$$

In particular, $e_{0^+}(\Psi) = \sup_{\mathbb{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}[-\Psi] \middle| h(\mathbb{Q} \,|\, \mathbb{P}) < +\infty \right\} = \rho_{\max}(\Psi)$ corresponds to the worst case risk measure.

Optimal Derivative Design

The program to be solved is the following :

$$R_{A,B}(X_T^A, X_T^B) = \inf_{F \in \mathcal{X}} \left\{ \rho_A \Big(X_T^A - F \Big) + \rho_B \Big(X_T^B + F \Big) \right\}.$$

⇒ We proceed in two steps :

- ⋄ Both agents assess their risk using the same family of risk measures,
- ⋄ Both agents use general risk measures.

⋆ **Optimal transaction between agents of the same family :**

Considering that both agents have γ-tolerant risk measures ρ_{γ_A} and ρ_{γ_B} from the same root risk measure ρ, the optimization program is written as :

$$R_{A,B}(X_T^A, X_T^B) = \inf_{F \in \mathcal{X}} \Big\{ \rho_{\gamma_A}(X_T^A - F) + \rho_{\gamma_B}(X_T^B + F) \Big\},$$

In this framework, the optimal risk transfer is consistent with Borch's theorem :

Theorem : *An optimal structure is given as :*

$$F^* = \frac{\gamma_B}{\gamma_A + \gamma_B} X_T^A - \frac{\gamma_A}{\gamma_A + \gamma_B} X_T^B \qquad \text{(to within a constant).}$$

⋆ **Optimal transaction in the general case :**

In a more general framework, we simply obtain a *necessary and sufficient condition* to have an optimal solution.

Optimal design problem with hedging opportunities

Framework

⋆ **Agent A** : at a future time T, agent A is exposed towards a non-tradable risk for an amount $\mathbf{X_T^A}$. She calls on an investor, agent B, to reduce her exposure by the *sale of a structured contract* **F**.

⋆ **Agent B** : she has her own exposure $\mathbf{X_T^B}$. She pays a premium $\boldsymbol{\pi}$ at time 0 and receives in exchange the structure F at time T.

⋆ To reduce their risk exposure, both agents can also invest in *financial markets* via two *convex sets* of bounded terminal gains associated with self-financing admissible investment strategies, $\boldsymbol{\mathcal{V}_T^{(A)}}$ and $\boldsymbol{\mathcal{V}_T^{(B)}}$.

⋆ Both agents assess their risk using a convex risk measure ($\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_B}$).

⇒ *The problem is to find the optimal structure* (F, π) *in such a framework.*

Hedging and Calibration strategies :

• **Dynamic (hedging) strategies :**

⇒ The *initial wealth* is a given data, invested in self-financing admissible portfolios, continuously rebalanced.

⇒ The *terminal gain* is characterized as the spread between the terminal wealth and the capitalized initial wealth.

• **Static (calibration) strategies :**

The strategies are based upon derivative instruments, sufficiently liquid so that the agents in the market agree on their price (possible bid-ask spread).

⇒ The *initial wealth* is not a given data, but the *market price* of the derivative.

⇒ The *terminal gain* is the spread between the derivative payoff and the forward (buyer's or seller's) price.

⇒ As underlined by the dual formulation of the problem, there are *consistency constraints* imposed by these strategies on the probability measures.

Both agents' objectives : *Both agents do not have the same goals.*

⋆ Agent A wants to determine the structure (F,π) as to minimize the risk measure of her terminal wealth :

$$\begin{aligned} \inf_{F\in\mathcal{X},\pi,\xi_A\in\mathcal{V}_T^{(A)}} \rho_A(X_T^A - F + \pi - \xi_A) &= \inf_{F\in\mathcal{X},\pi} \inf_{\xi_A\in\mathcal{V}_T^{(A)}} \rho_A(X_T^A - F + \pi - \xi_A) \\ &= \boxed{\inf_{F,\pi} \rho_A^m(X_T^A - F + \pi).} \end{aligned}$$

⋆ Agent B wants to improve her risk measure by doing the F-transaction. Her interest in doing the transaction may be written as :

$$\inf_{\xi_B\in\mathcal{V}_T^{(B)}} \rho_B(X_T^B + F - \pi - \xi_B) \le \inf_{\xi_B\in\mathcal{V}_T^{(B)}} \rho_B(X_T^B - \xi_B)$$

or equivalently, using the market modified risk measure ρ_B^m :

$$\boxed{\rho_B^m(X_T^B + F - \pi) \le \rho_B^m(X_T^B).}$$

Optimal pricing rule and residual risk measure

⋆ The optimal pricing rule is simply obtained by binding the constraint of Agent B and using the cash invariance property :

$$\boxed{\pi^*(F) = \rho_B^m(X_T^B) - \rho_B^m(X_T^B + F).}$$

⋆ The program

$$\inf_{F,\pi} \rho_A^m(X_T^A - F + \pi) \qquad \text{subject to} \quad \rho_B^m(X_T^B + F - \pi) \leq \rho_B^m(X_T^B)$$

may be reduced to (to within the constant $\rho_B^m(X_T^B)$)

$$\boxed{\begin{aligned} R_{AB}^m(X_T^A + X_T^B) &\overset{def}{=} \inf_F(\rho_A^m(X_T^A - F) + \rho_B^m(X_T^B + F)) \\ &= \rho_A^m \Box \rho_B^m(X_T^A + X_T^B) \\ &= \rho_A \Box \nu^{\mathcal{V}_T^{(A)}} \Box \rho_B \Box \nu^{\mathcal{V}_T^{(B)}}(X_T^A + X_T^B) \\ &= \inf_{\xi \in \mathcal{V}_T^{(A)} + \mathcal{V}_T^{(B)}} \rho_A \Box \rho_B \left(X_T^A + X_T^B - \xi\right). \end{aligned}}$$

⇒ Using the previous results, R_{AB}^m is a convex risk measure.

Characterization of the optimal structure - Dilated framework

Proposition : *Assume that both agents have* <u>*dilated risk measures*</u>*,* ρ_A *and* ρ_B*, associated with the respective risk tolerance coefficients* γ_A *and* γ_B*.*
(i) If they have the same access to the financial market, then an optimal structure is given by :

$$\boxed{F^* = \frac{\gamma_B}{\gamma_A+\gamma_B} X_T^A - \frac{\gamma_A}{\gamma_A+\gamma_B} X_T^B.}$$

(ii) If they do not have the same access to the financial market and if $\xi^* = \eta_A^* + \eta_B^*$ *is an optimal solution of the Program* $\inf_{\xi \in \mathcal{V}_T^{(A+B)}} \rho_A \Box \rho_B \left(X_T^A + X_T^B - \xi \right)$ *with* $\eta_A^* \in \mathcal{V}_T^{(A)}$ *and* $\eta_B^* \in \mathcal{V}_T^{(B)}$*. Then*

$$\boxed{F^* = \frac{\gamma_B}{\gamma_A+\gamma_B} X_T^A - \frac{\gamma_A}{\gamma_A+\gamma_B} X_T^B - \frac{\gamma_B}{\gamma_A+\gamma_B} \eta_A^* + \frac{\gamma_A}{\gamma_A+\gamma_B} \eta_B^*}$$

is an optimal structure.
Moreover, η_B^* *is an optimal hedging portfolio of* $X_T^B + F^*$ *for agent* B *and* η_A^* *is an optimal hedging portfolio of* $X_T^A - F^*$ *for agent* A*.*

In a more general framework, we simply obtain a *necessary and sufficient condition* to have an optimal solution.

⇒ Obtaining an explicit characterization of this transfer requires some technical methods. The use of *dynamic programing techniques* (BSDEs), may help to study risk measures defined by their local specifications.

Dynamic Framework

Localization of convex risk measures : An axiomatic approach

We now consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t; t \geq 0))$ where $(\mathcal{F}_t)$ is a Brownian filtration $(\mathcal{F}_t = \sigma(W_s; 0 \leq s \leq t); t \geq 0)$.

The idea behind our approach is to find a trade-off between static and very abstract dynamic risk measures as to obtain *tractable risk measures*.

⋆ **Dynamic operators**

A dynamic L^2-operator (L^∞-operator) $\mathcal{Y}$ with respect to $\Big(\mathcal{F}_t; t \geq 0\Big)$ is a family of continuous semi-martingales which maps, for any bounded stopping time T, a $L^2(\mathcal{F}_T)$ (resp. $L^\infty(\mathcal{F}_T)$) -variable ξ_T onto a process $\Big(\mathcal{Y}_t(\xi_T); t \in [0, T]\Big)$.

⋆ **Dynamic convex risk measures**

Proposition : A *dynamic convex risk measure* is a dynamic operator satisfying the following axioms :

For any bounded stopping times $S \leq T \leq U$, for any (ξ_T^1, ξ_T^2) :

- *Convexity* : $\forall 0 \leq \lambda \leq 1,\ \mathcal{Y}_S(\lambda \xi_T^1 + (1-\lambda)\xi_T^2) \leq \lambda \mathcal{Y}_S(\xi_T^1) + (1-\lambda)\mathcal{Y}_S(\xi_T^2)\ a.s.$
- *Decreasing Monotonicity* : If $\xi_T^1 \geq \xi_T^2$ *a.s.*, $\mathcal{Y}_S(\xi_T^1) \leq \mathcal{Y}_S(\xi_T^2) \quad a.s.$
- *Translation invariance (-)* : $\forall \eta_S \in \mathcal{F}_S$, $\forall \xi_T$, $\mathcal{Y}_S(\xi_T + \eta_S) = \mathcal{Y}_S(\xi_T) - \eta_S \quad a.s.$

- *Time-consistency (-)* : $\forall \xi_U$, $\mathcal{Y}_S(\xi_U) = \mathcal{Y}_S(-\mathcal{Y}_T(\xi_U))$ *a.s.*
- *No-arbitrage* : $\xi_T^1 \geq \xi_T^2$, $\mathcal{Y}_S(\xi_T^1) = \mathcal{Y}_S(\xi_T^2)$ on $A_S = \{S < T\} \Rightarrow\ \xi_T^1 = \xi_T^2\ a.s.$ on A_S.

⋆ **Relationship with consistent convex price system and non-linear expectation**

The dynamic convex risk measures are closely related to :

• *Consistent convex (forward) price system*, introduced by El Karoui and Quenez (1996), defined as a dynamic operator being convex, increasing, time-consistent (+) and without arbitrage.

• *Non-linear expectation* introduced by Peng (1997), defined as a dynamic operator being convex, increasing, time-consistent (+), translation-invariant (+) and satisfying another property close to no-arbitrage.

Dynamic entropic risk measure :

It is defined as :

$$\boxed{e_{\gamma,t}(X) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma}X\right)/\Im_t\right)}$$

is a typical dynamic convex risk measure.

Relationships with BSDEs :

Moreover, it is possible to relate it to BSDE since $(e_{\gamma,t}(X); t \in [0,T])$ is solution of the BSDE with the quadratic coefficient $g(t,z) = \frac{1}{2\gamma}\|z\|^2$:

$$\boxed{-de_{\gamma,t}(X) = \frac{1}{2\gamma}\|z_t\|^2 dt - \langle z_t, dW_t\rangle \qquad \text{with the terminal condition } e_{\gamma,T}(X) = -X}$$

$\Rightarrow$ The idea is then to relate dynamic convex risk measures with BSDEs.

Dynamic convex risk measures and BSDEs

The BSDE (g, ξ_T) :

$$-dY_t = g(t, Y_t, Z_t)dt - < Z_t, dW_t > , \quad Y_T = \xi_T$$

has a (maximal, unique ?) solution (Y_t, Z_t) under some conditions imposed on the coefficient g :

- ⋄ Uniformly Lipschitz (H1) (and $\xi_T \in L^2$),
- ⋄ Continuous with linear growth (H2) (and $\xi_T \in L^2$),
- ⋄ Continous with quadratic growth (H3) (and $\xi_T \in L^\infty$).

⋆ Let g be a standard coefficient. The g-dynamic operator, denoted by $\mathcal{Y}^g$, is such that $\mathcal{Y}^g_t(\xi_T)$ is the maximal solution of the BSDE (g, ξ_T) :

$$-dY_t = g(t, Y_t, Z_t)dt - < Z_t, dW_t > , \quad Y_T = \xi_T$$

Theorem : *Let $\mathcal{Y}^g$ be the g-dynamic operator.*

⋆ *Then, $\mathcal{Y}^g$ is increasing monotonic, time-consistent (+) and arbitrage-free.*

⋆ *Moreover,*

1. *$\mathcal{Y}^g$ is translation invariant (+) if and only if g does not depend on y.*
2. *$\mathcal{Y}^g$ is homogeneous if and only if g is homogeneous ;*

⋆ *For properties related to the order, the following implications simply hold :*

1. *If g is convex, then $\mathcal{Y}^g$ is convex.*
2. *If $g^1 \leq g^2$, then $\mathcal{Y}^{g^1} \leq \mathcal{Y}^{g^2}$.*

Therefore, if g is a convex coefficient depending only on z, $\mathcal{R}^g(\xi_T) \equiv \mathcal{Y}^g(-\xi_T)$ is a dynamic convex risk measure, called g-conditional risk measure.

Infinitesimal risk measure :

$$\boxed{-d\mathcal{R}_t^g = g(t, Z_t)dt - < Z_t, dW_t > , \quad \mathcal{R}_T^g = -\xi_T}$$

$\Rightarrow$ The coefficient of any g-conditional risk measure $\mathcal{R}^g$ can be naturally interpreted as the *infinitesimal risk measure* over a time interval $[t, t+dt]$ as :

$$\mathbb{E}_{\mathbb{P}}[d\mathcal{R}_t^g|\mathcal{F}_t] = -g(t, Z_t)dt,$$

where Z_t is the local volatility of the g-conditional risk measure :

$$\mathbb{V}(d\mathcal{R}_t^g|\mathcal{F}_t) = |Z|_t^2 dt$$

$\Rightarrow$ Choosing carefully the coefficient g enables to generate g-conditional risk measures that are *locally compatible* with the views and practice of the different agents in the market.

Dual representation of g-conditional risk measures

$\star$ A g-conditional risk measure $\mathcal{R}^g$ is said to have a *dual representation* if there exists a set $\mathcal{A}$ of admissible controls such that for any bounded s.t. $S \leq T$:

$$\boxed{\mathcal{R}^g_S(\xi_T) = \operatorname{ess\,sup}_{\mu \in \mathcal{A}} \mathbb{E}_{\mathbb{Q}^\mu}\Big[- \xi_T - \int_S^T G(t, \mu_t) dt \Big| \mathcal{F}_S \Big]}$$

where $G(t,.)$ is the polar function of $g(t,.)$.

$\star$ The problem is then to characterize the set $\mathcal{A}$ according to the conditions imposed on the coefficient g :

$\Rightarrow$ Under (H1) (uniformly Lipschitz) (solved by El Karoui-Peng-Quenez (1997)), $\mathcal{A}$ is the space of adapted processes μ bounded by the Lipschitz constant and $\mathbb{Q}^\mu$ is the equivalent probability measure with density $\Gamma^\mu_T : d\Gamma^\mu_t = \Gamma^\mu_t \mu^*_t dW_t, \quad \Gamma^\mu_0 = 1.$

$\Rightarrow$ Under (H3) (quadratic growth), $\mathcal{A}$ is the space of BMO($\mathbb{P}$)-processes μ and $\mathbb{Q}^\mu$ is defined as above.

Possibly infinite risk measures

⋆ We can consider as "acceptable" the investment in an admissible portfolio and as infinitely risky any non-replicable terminal payoff.

⋆ More precisely, for any U in a convex space $\mathcal{K}$, let us consider :

$$V_t^U = V_0^U + \int_0^t U_s \, \lambda_s dt + \int_0^t U_s \, dW_s.$$

⋆ We define a dynamic risk measure $\mathcal{L}_t^{\mathcal{K}}(\xi_T)$ as :

$$\boxed{\mathcal{L}_t^{\mathcal{K}}(\xi_T) = -V_t^U \quad \text{if } \xi_T = V_T^U \quad , \quad \mathcal{L}_t^{\mathcal{K}}(\xi_T) = +\infty \quad \text{otherwise.}}$$

V_t^U may be in general interpreted as the *arbitrage price at time t of the attainable contingent claim V_T^U*.

⋆ The dual representation is associated with the support function $\sigma_{\mathcal{K}_t}$ of the convex set $\mathcal{K}_t$:

$$\mathcal{L}_t^{\mathcal{K}}(\xi_T) = \text{esssup}_\mu \mathbb{E}_{\mathbb{Q}^\mu}\Big[-\xi_T - \int_t^T \sigma_{\mathcal{K}_t}(\mu_t + \lambda_t) ds \Big| \mathcal{F}_t \Big].$$

$\star$ **Dilated g-conditional risk measure** : a g-conditional risk measure, $\mathcal{R}^g_\gamma$ is the *γ-dilated of* $\mathcal{R}^g$ if and only if g_γ is γ-dilated of g :

$$g_\gamma(t,z) = \gamma g\Big(t, \frac{1}{\gamma} z\Big).$$

$\star$ **Conservative risk measure and super-pricing** : the *g-conservative risk measure* is associated with the coefficient g_{0+} defined by $g_{0+}(t,z) := \lim_{\gamma\downarrow 0} g_\gamma(t,z)$.
$\mathcal{R}^g_{0+}$ is characterized by :

$$\mathcal{R}^g_{0+,t}(\xi_T) = \operatorname{ess\,sup}_{\mu\in\mathcal{A}\cap\mathrm{Dom}(G)} \mathbb{E}_{\mathbb{Q}^\mu}\Big[-\xi_T \Big| \mathcal{F}_t\Big].$$

It corresponds to the equivalent of the super-pricing rule of $-\xi_T$.

$\star$ **Marginal risk measure** : the *g-marginal risk measure* is associated with the coefficient g_∞ defined as the limit of g_γ when $\gamma \to \infty$.
$\mathcal{R}^g_\infty$ is characterized by :

$$\mathcal{R}^g_{\infty,t}(\xi_T) = \operatorname{ess\,sup}_{\mu\in\mathcal{A}} \Big\{ \mathbb{E}_{\mathbb{Q}^\mu}\Big[-\xi_T \Big| \mathcal{F}_t\Big] \Big| \, G(u,\mu_u) = 0,\ \forall u \geq t,\ a.s. \Big\}.$$

In some cases, $-\mathcal{R}^g_\infty$ is a linear pricing rule and can be seen as an extension of the Davis price.

Inf-convolution of g-conditional risk measures

We now come back to our inf-convolution problem.

⇒ We denote the g-conditional risk measures of both agents by $\mathcal{R}^A$ and $\mathcal{R}^B$ (the associated convex coefficients are respectively g^A and g^B).

⇒ We then introduce :

⋆ the BSDE associated with the coefficient $(g^A \square g^B)$:

$-d\mathcal{R}_t^{A,B}(X) = \left(g^A \square g^B\right)(t, Z_t)\, dt - \langle Z_t, dW_t \rangle$ with terminal condition $\mathcal{R}_T^{A,B}(X) = -X$

⋆ the inf-convolution of $\mathcal{R}_t^A$ and $\mathcal{R}_t^B$ defined by

$$\left(\mathcal{R}^A \square \mathcal{R}^B\right)_t (X) = \inf_F \left\{\mathcal{R}_t^A (X - F) + \mathcal{R}_t^B (F)\right\}.$$

$(\mathcal{R}_t^{A,B}(X), Z_t)$ is the solution of $-d\mathcal{R}_t^{A,B}(X) = (g^A \square g^B)(t, Z_t)dt - \langle z_t, dW_t \rangle$, $\mathcal{R}_T^{A,B}(X) = -X$

Theorem : *The following results hold :*

(i) For any $t \in [0,T]$ and for any F such that both $\mathcal{R}_t^A(X-F)$ and $\mathcal{R}_t^B(F)$ are well defined : $\mathcal{R}_t^{A,B}(X) \leq \mathcal{R}_t^A(X-F) + \mathcal{R}_t^B(F) \quad \mathbb{P}$ *a.s.*

(ii) If there exists an admissible $\widehat{Z}_t^B$ such that $\forall t \geq 0$:

$$g^A \square g^B (t, Z_t) = \inf_{Z_t^B} \left\{ g^A \left(t, Z_t - Z_t^B\right) + g^B \left(t, Z_t^B\right) \right\} = g^A \left(t, Z_t - \widehat{Z}_t^B\right) + g^B \left(t, \widehat{Z}_t^B\right)$$

then $\forall t \geq 0 \qquad \mathcal{R}_t^{A,B}(X) = \left(\mathcal{R}^A \square \mathcal{R}^B\right)_t (X) \quad \mathbb{P}$ *a.s.*

Moreover,

$$F^* = \int_0^T g^B \left(t, \widehat{Z}_t^B\right) dt - \int_0^T \left\langle \widehat{Z}_t^B, dW_t \right\rangle$$

is an optimal solution for the inf-convolution problem

$$\left(\mathcal{R}^A \square \mathcal{R}^B\right)_t (X) = \inf_F \left\{ \mathcal{R}_t^A (X-F) + \mathcal{R}_t^B (F) \right\}.$$

Non-speculative logic

We now assume that none of the two agents bear any initial risk.

Corollary :
Assume that $g^A(t,0) = g^B(t,0) = 0$*, then :*
(i) $F^* \equiv 0$ *is an optimal solution for the inf-convolution problem*

$$\left(\mathcal{R}^A \Box \mathcal{R}^B\right)_t (0) = \inf_F \left\{ \mathcal{R}_t^A(-F) + \mathcal{R}_t^B(F) \right\}.$$

(ii) *If both drivers* g^A *and* g^B *are strictly convex, then* $F^* \equiv 0$ *is the unique optimal solution for the inf-convolution problem.*

$\Rightarrow$ In this case the logic of the transaction is *non-speculative* since the issuer has an interest to sell a structure if and only if she is initially exposed.