
Travelling breathers in Klein-Gordon chains

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Introduction

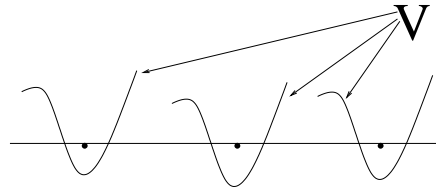
1 – Introduction

Introduction

We consider the dynamics of a one-dimensional lattice of nonlinear oscillators described by the following system (Klein-Gordon system)

$$\frac{d^2 x_n}{d\tau^2} + V'(x_n) = \gamma(x_{n+1} + x_{n-1} - 2x_n), \quad n \in \mathbb{Z}$$

where



- x_n : displacement of the n th particle from an equilibrium position
- V : smooth on-site potential, $V'(0) = 0$, $V''(0) = 1$.
- $\gamma > 0$: coupling constant

We consider solutions of (1) satisfying

$$x_n(\tau) = x_{n-p}(\tau - T)$$

where T is a given constant and $p \geq 2$ integer. Here, we consider $p = 2$.

Definition of travelling breathers :

$$(2) + \lim_{\tau \rightarrow \pm\infty} x_n(\tau) = 0 .$$

REFERENCES :

- Presentation based on : Sire-James, *Travelling breathers in Klein-Gordon chains*, CRAS, 2004, James-Sire, *Travelling breathers with exponentially small tails in a chain of nonlinear oscillators*, *Commun. Maths. Phys.*, 2005, Sire-James, *Numerical computation of travelling breathers in Klein-Gordon chains*, *Physica D*, 2005
- *Static breathers* : MacKay-Aubry (anti-continuous limit, 1994), James (center manifold reduction, 2003), Aubry-Kopidakis-Kadelburg (variational approach, 2001), Arioli-Gazzola (variational approach for FPU chains without phonons, 1996).
- *Travelling wave solutions* : Iooss-Kirchgässner (nanopterons in Klein-Gordon, 2000) FPU : Friesecke-Wattis (1994), Smets-Willem (1997), Friesecke-Pego (1999/2002/2004), Iooss (2000).

Formulation of the problem

2 – Formulation of the problem

Formulation of the problem

Solutions satisfying $x_n(\tau) = x_{n-2}(\tau - T)$ are determined by

$$\frac{d^2}{d\tau^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} V'(x_1) \\ V'(x_2) \end{bmatrix} = \gamma \begin{bmatrix} x_2(\tau) - 2x_1(\tau) + x_2(\tau + T) \\ x_1(\tau) - 2x_2(\tau) + x_1(\tau - T) \end{bmatrix}.$$

We rescale (3) using $t = \frac{\tau}{T}$ and consider the new variables $(u_1(t), u_2(t)) = (x_1(\tau), x_2(\tau + \frac{T}{2}))$. This yields

$$\begin{aligned} x_n(t) &= u_1\left(\frac{t}{T} - \frac{n-1}{2}\right) \text{ if } n \text{ is odd,} \\ x_n(t) &= u_2\left(\frac{t}{T} - \frac{n-1}{2}\right) \text{ if } n \text{ is even.} \end{aligned}$$

With this change of variables, we have

$$\frac{d^2}{dt^2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + T^2 \begin{bmatrix} V'(u_1) \\ V'(u_2) \end{bmatrix} = \gamma T^2 \begin{bmatrix} u_2(t - \frac{1}{2}) - 2u_1(t) + u_2(t + \frac{1}{2}) \\ u_1(t + \frac{1}{2}) - 2u_2(t) + u_1(t - \frac{1}{2}) \end{bmatrix}.$$

Formulation of the problem

We can write (5) as an evolution problem in a Banach space.

We set $U = (u_1, u_2, \dot{u}_1, \dot{u}_2, X_1(t, v), X_2(t, v))$ where $v \in [-1/2, 1/2]$ and

$$X_1(t, v) = u_1(t + v), X_2(t, v) = u_2(t + v).$$

$$\partial_t u_1 = \dot{u}_1, \partial_t u_2 = \dot{u}_2$$

$$\partial_t \dot{u}_1 = -T^2 V'(u_1) + \gamma T^2 (X_2(t, 1/2) - 2u_1 + X_2(t, -1/2))$$

$$\partial_t \dot{u}_2 = -T^2 V'(u_2) + \gamma T^2 (X_1(t, 1/2) - 2u_2 + X_1(t, -1/2))$$

$$\partial_t X_1 = \partial_v X_1, \partial_t X_2 = \partial_v X_2$$

with conditions

$$X_1(t, 0) = u_1(t), X_2(t, 0) = u_2(t)$$

Formulation of the problem

CONCISE NOTATION :

$$\frac{dU}{dt} = LU + F(U)$$

We have $U(t) \in \mathbb{D}$ and $\frac{dU}{dt}(t) \in \mathbb{H}$ with

$$\mathbb{H} = \mathbb{R}^4 \times (C^0[-1/2, 1/2])^2$$

$$\mathbb{D} = \{U \in \mathbb{R}^4 \times (C^1[-1/2, 1/2])^2 / X_1(0) = u_1, X_2(0) = u_2\}.$$

The linear operator L maps \mathbb{D} into \mathbb{H} continuously,
 $F : \mathbb{D} \rightarrow \mathbb{H}$ is smooth with $F(U) = O(\|U\|_{\mathbb{D}}^2)$ and

$$F(U) = T^2(0, 0, u_1 - V'(u_1), u_2 - V'(u_2), 0, 0),$$

Formulation of the problem

- Reversibility symmetry R

$$R(u_1, u_2, \xi_1, \xi_2, X_1(v), X_2(v)) = (u_1, u_2, -\xi_1, -\xi_2, X_1(-v), X_2(-v))$$

$U(t)$ is a solution $\rightarrow RU(-t)$ is a solution

- Permutational symmetry S

$$S(u_1, u_2, \xi_1, \xi_2, X_1, X_2) = (u_2, u_1, \xi_2, \xi_1, X_2, X_1)$$

$U(t)$ is a solution $\rightarrow SU(t)$ is a solution

3 – Spectrum of the linearized evolution operator

Spectrum of the linearized evolution operator

The spectrum of L consists in isolated eigenvalues σ with finite multiplicities.

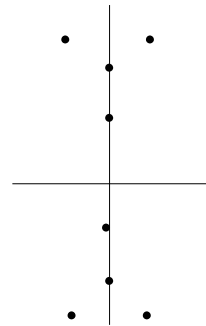
Solving $LU = \sigma U$ leads to the dispersion relation

$$N(\sigma, T, \gamma) := (\sigma^2 + T^2(1 + 2\gamma))^2 - 4(\gamma T^2)^2 \cosh^2(\sigma/2) = 0. \quad ($$

For the central part of the spectrum ($\sigma = iq$), the dispersion relation reads

$$(-q^2 + T^2(1 + 2\gamma))^2 = 4(\gamma T^2)^2 \cos^2(q/2). \quad ($$

Considered bifurcation :



We define Δ as the subset of Γ (bifurcation curve) such that the central part of the spectrum is $\Sigma_0 = \{\pm iq_1, \pm iq_2, \pm iq_0\}$ where $\pm iq_0$ is a pair of non semi-simple dou eigenvalues and $\pm iq_1, \pm iq_2$ two pairs of simple ones.

Spectrum of the linearized evolution operator

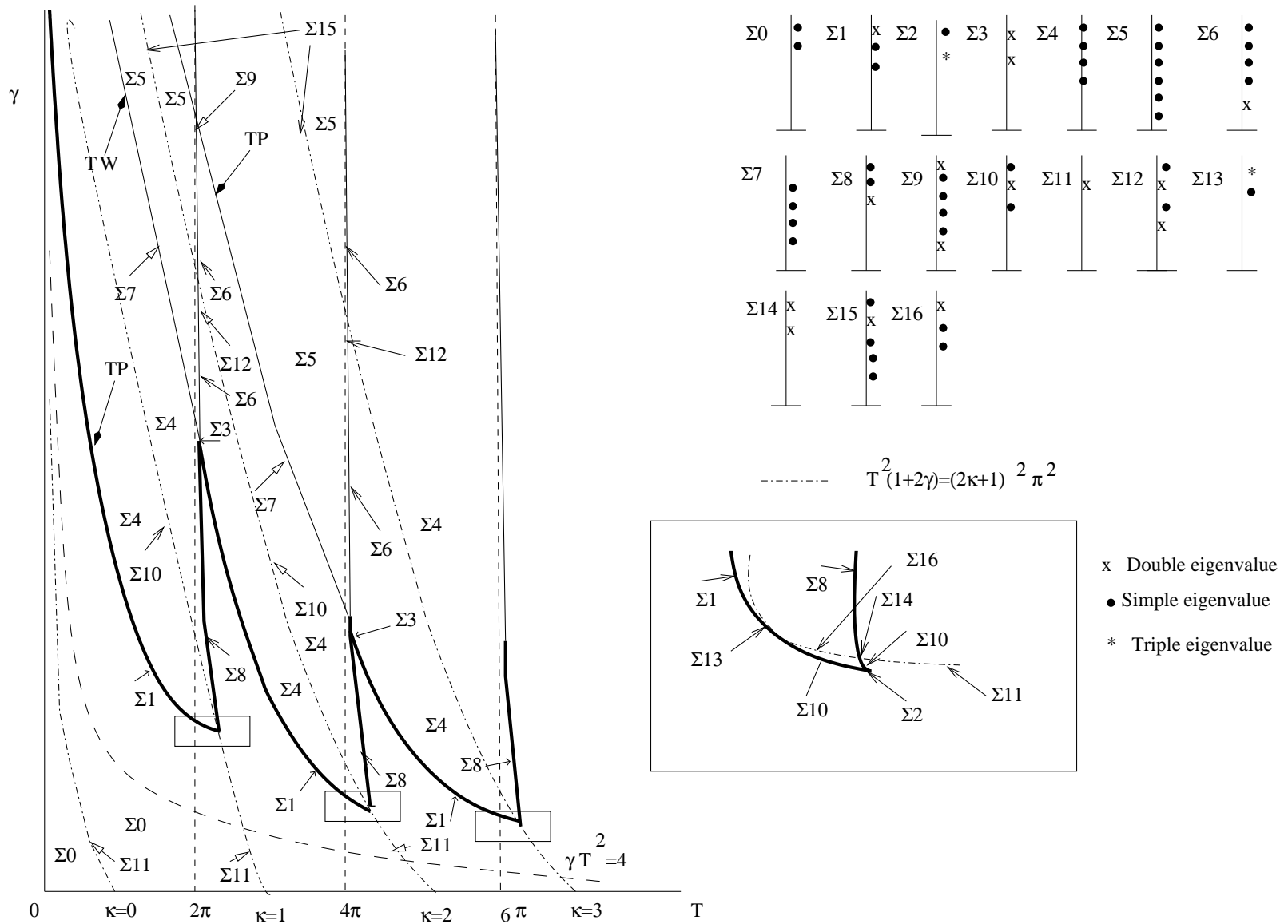
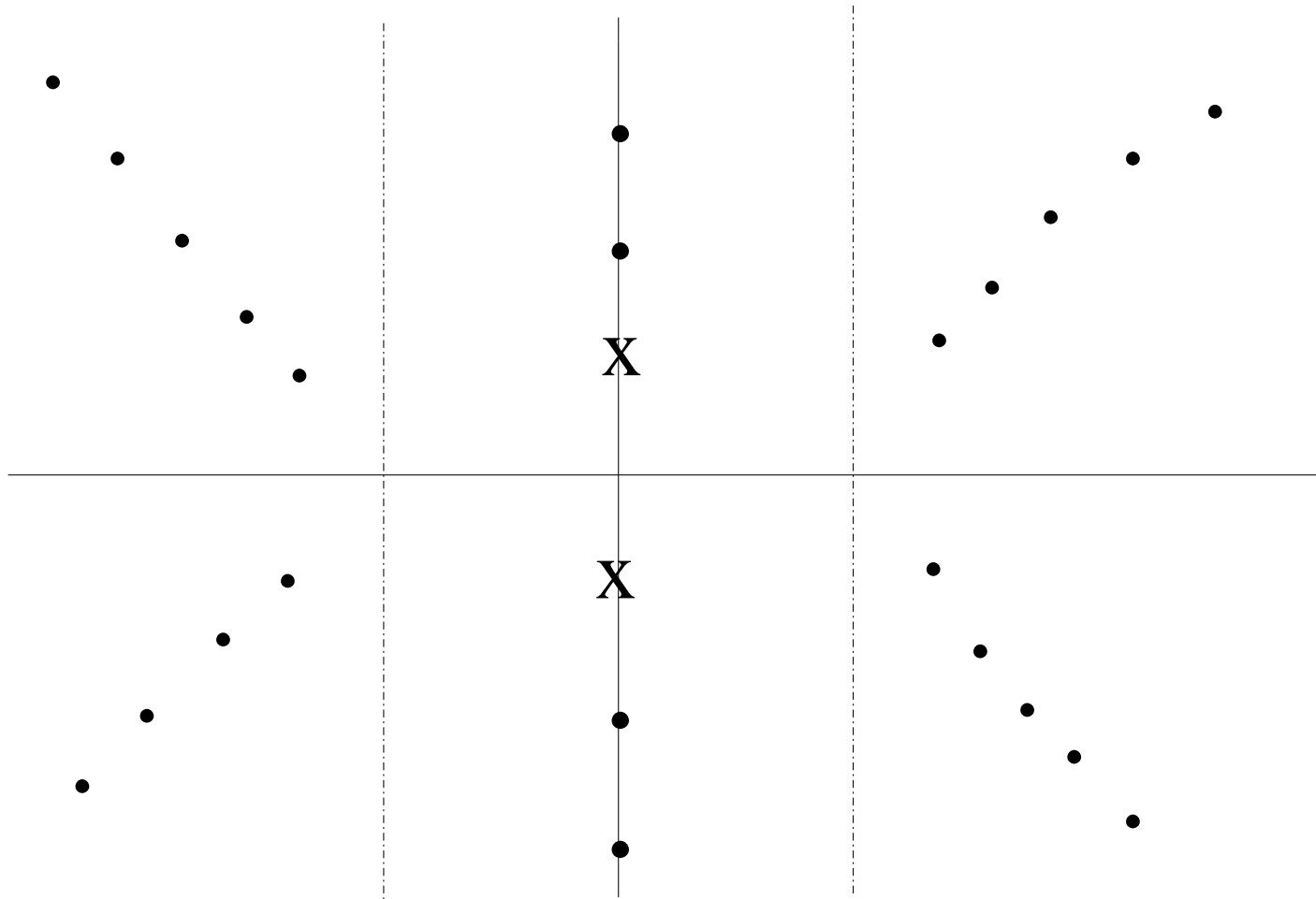


FIG. 1: Purely imaginary spectrum of L

4 – Regularity problem and reduction

Regularity problem and reduction



Regularity problem and reduction

We fix $(\gamma, T) = (\gamma_0, T_0) \in \Delta$.

- P : projection on the 8-dimensional invariant subspace corresponding to $\{\pm iq_0, \pm iq_1, \pm iq_2\}$ (central subspace),
- $\mathbb{D}_h = (\mathbb{I} - P)\mathbb{D}, \mathbb{D}_c = P\mathbb{D}, U_h = (\mathbb{I} - P)U, U_c = PU$
- **CENTER MANIFOLD REDUCTION of the nonlinear problem :**
For $(\gamma, T) \approx (\gamma_0, T_0)$, small amplitude solutions satisfy

$$U_h(t) = \psi(U_c(t), \gamma, T)$$

where $\psi : \mathbb{D}_c \times \mathbb{R}^2 \mapsto \mathbb{D}_h$ is smooth.

- Finite dimensional problem (eight-dimensional)

$$\frac{dU_c}{dt} = P(L(U_c + \psi(U_c, \gamma, T)) + F(U_c + \psi(U_c, \gamma, T))). \quad (12)$$

(12) inherits all the symmetries (reversible under R and equivariant under S).

Regularity problem and reduction

Affine problem on the hyperbolic subspace $\mathbb{H}_h = (\mathbb{I} - P)\mathbb{H}$

$$\frac{dU_h}{dt} = LU_h + F_h(t) \quad ($$

where $F(t) = (0, 0, f_1(t), f_2(t), 0, 0)$ lies in the range of the nonlinear operator. Using Fourier Transform and a regularity lemma by looss and Kirchgässner, we prove

Lemma :

Assume $F = (0, 0, f_1, f_2, 0, 0)$ and $f_1, f_2 \in C_b^0(\mathbb{R})$. Then the affine linear system

$$\frac{dU_h}{dt} = LU_h + F_h(t)$$

has a unique solution $U_h \in C_b^0(\mathbb{D}_h) \cap C_b^1(\mathbb{H}_h)$ and the map $K_h : C_b^0(\mathbb{R}^2) \rightarrow C_b^0(\mathbb{D}_h)$, $(f_1, f_2) \mapsto U_h$ is bounded.

REMARK : the form of F_h is important to derive this regularity result.

5 – Study of the reduced equation

5.1 – Normal form

Study of the reduced equation

By a polynomial change of variables $U_c = P_{\gamma, T}(A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D})$, we obtain (normal form at order 3)

$$\frac{dA}{dt} = iq_0 A + B + iA\mathcal{P}(|A|^2, |C|^2, |D|^2, w) + \text{h.o.t}$$

$$\frac{dB}{dt} = iq_0 B + iB\mathcal{P}(|A|^2, |C|^2, |D|^2, w) + A\mathcal{S}(|A|^2, |C|^2, |D|^2, w) + \text{h.o.t}$$

$$\frac{dC}{dt} = iq_1 C + iC\mathcal{Q}(|A|^2, |C|^2, |D|^2, w) + \text{h.o.t}$$

$$\frac{dD}{dt} = iq_2 D + iD\mathcal{T}(|A|^2, |C|^2, |D|^2, w) + \text{h.o.t}$$

where $w = i(A\bar{B} - \bar{A}B)$ and $\mathcal{P}, \mathcal{S}, \mathcal{Q}, \mathcal{T}$ are affine functions of $|A|^2, |C|^2, |D|^2, w$,

Study of the reduced equation

TRUNCATED SYSTEM :

$|C|^2, |D|^2$ are first integrals \rightarrow 1 : 1 resonance with reversibility (Iooss-Pérouème 1993).

Homoclinic orbits to 0 : $A = r e^{i(q_0 t + \psi)}, B = r' e^{i(q_0 t + \psi)}$ satisfying

$$r'' = s_1(\gamma, T)r + s_2(\gamma, T)r^3,$$

$$\psi' = p_1(\gamma, T) + p_2(\gamma, T)r^2.$$

CRUCIAL BIFURCATION COEFFICIENT :

$$\left(2 - \frac{q_0}{\tan(q_0/2)}\right) s_2 = T_0^2(-6\beta + 8\alpha^2) - \frac{4\alpha^2 T_0^2}{2\gamma_0 T_0^2 \cos(q_0) - T_0^2(1 + 2\gamma_0) + 4q_0^2}$$

with $V(x) = \frac{1}{2}x^2 + \frac{\alpha}{3}x^3 + \frac{\beta}{4}x^4 + \text{h.o.t.}$

Study of the reduced equation

5.2 – Description of the small amplitude solutions

Study of the reduced equation

We choose $(\gamma, T) \approx (\gamma_0, T_0)$ ($(\gamma_0, T_0) \in \Delta_0$), in such a way that the linearized operator L has four symmetric eigenvalues close to $\pm iq_0$ and having non zero real part ($s_1(\gamma, T) > 0, s_1(\gamma, T) \approx 0$).

- $s_2(\gamma_0, T_0) < 0$: existence of homoclinic orbits to 0, to periodic or quasi-periodic orbits for the **truncated normal form**.

Difficulty : **PERSISTENCE** for the full reduced equation.

- Existence of small amplitude travelling wave solutions x_n for $(T_0, \gamma_0) \in \Gamma_{2k}$ ($C =$ (Iooss and Kirchgässner)
→ **PERSISTENCE** of *S-invariant* reversible orbits, homoclinic to periodic orbits of exponentially small amplitudes above a critical tail size $|D|_{crit} = O(e^{-c/\mu^{1/2}})$ when $\mu = |T - T_0|, \gamma = \gamma_0$ fixed. Follows from Lombardi (2000).
- $(T_0, \gamma_0) \in \Gamma_{2k+1}$: open problem for the persistence of homoclinic orbits to quasi-periodic orbits. These solutions should correspond to travelling breathers with quasi-periodic oscillatory tail.

5.3 – Persistence result in a symmetric case

Study of the reduced equation

ASSUMPTION: the potential V is even.

$-S$ is a symmetry of the system. Fixed points of $-S$ correspond to solutions satisfying

$$x_{n+1}(\tau) = -x_n\left(\tau - \frac{T}{2}\right). \quad (1)$$

On $\text{Fix}(-S)$, one has $D = 0$ in the normal form.

Persistence of small amplitude reversible homoclinic orbits to periodic ones above a critical tail size $|C|_{crit} = O(e^{-c/\mu^{1/2}})$ ($\mu = |T - T_0|$, $\gamma = \gamma_0$ fixed). Travelling breathers + tail :

$$x_n(\tau) = (-1)^n [A + C] \left(\frac{\tau}{T} - \frac{n-1}{2} \right) + c.c + h.o.t.$$

Their principal part (excluding the tail) coincides with formal leading order solutions obtained by Remoissenet (1986) :

$$x_n(\tau) \approx \delta^{1/2} \hat{A}(\delta^{1/2} (n - v_g \tau)) e^{i(k_0 n - \omega_0 \tau)} + c.c$$

where $\delta = T - T_0 \approx 0$, $k_0 = \frac{q_0}{2} - \pi$, $\omega_0 = \frac{q_0}{T_0}$, $v_g = \frac{2}{T_0}$ and \hat{A} has the form

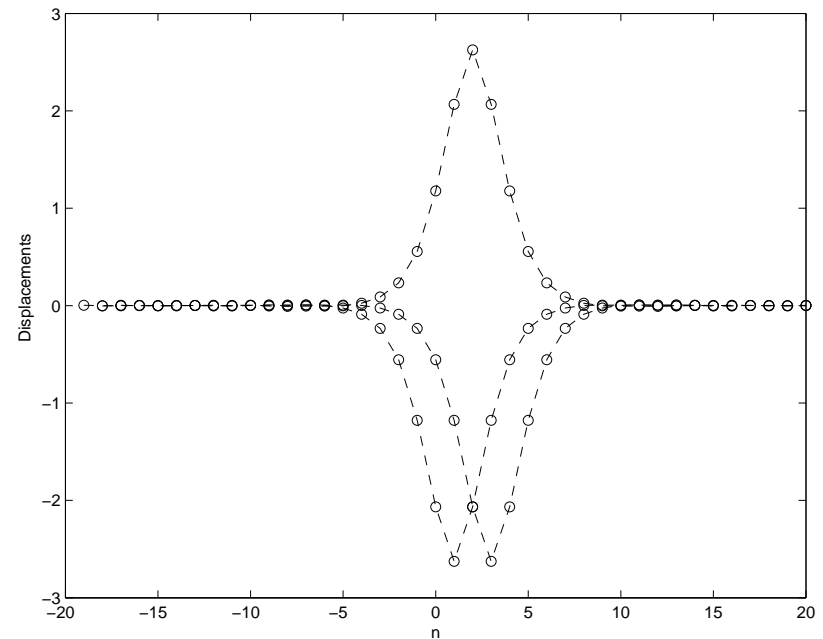
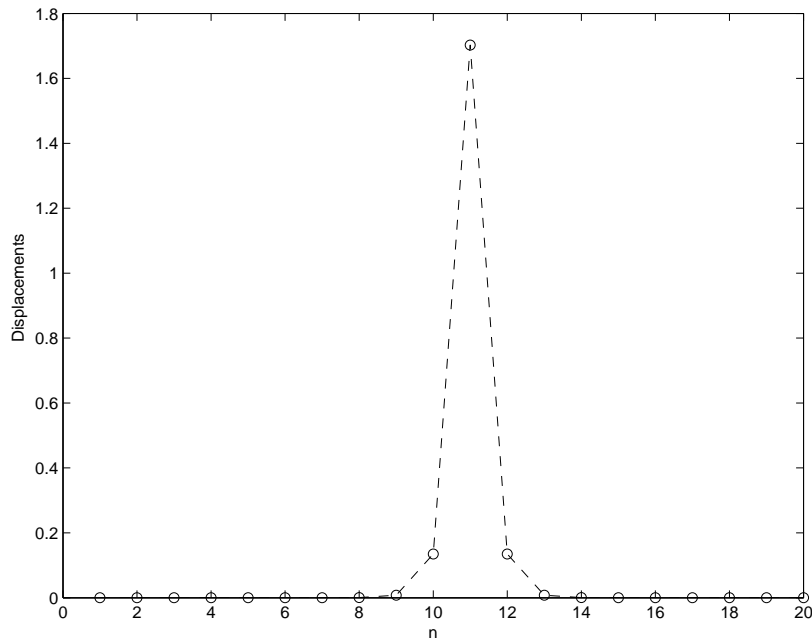
$$\hat{A}(\xi) = C_1 (\cosh(c_2 \xi))^{-1}.$$

Study of the reduced equation

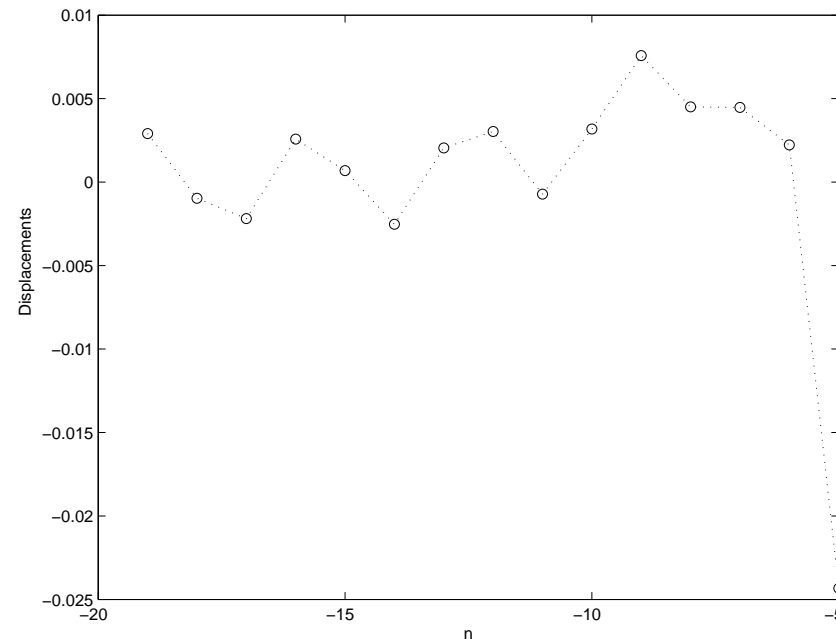
METHOD : Finite difference scheme on the advance-delay differential equations

→ Resolution of a nonlinear algebraic system of equations

Static Breather ($V(x) = \frac{1}{2}(e^{-x} - 1)^2$) Travelling Breather ($V(x) = 1 - \cos(x)$)



Study of the reduced equation



Interpretation : to be coherent, the breather needs to interact with resonant phonons.

Underlying phenomenon : phenomena beyond all algebraic orders with respect to the size of oscillations at the center → existence of an oscillating tail