Kinetic limit for wave propagation in a harmonic crystal with small random mass perturbations

DERIVATION OF A PHONON BOLTZMANN EQUATION

Jani Lukkarinen

with Herbert Spohn

Zentrum Mathematik

Technische Universität München

Outline

- Classical harmonic crystal

  . . . with small random mass perturbations

- Linear phonon Boltzmann equation

- Theorem (kinetic limit of the Wigner transform)

- Proof:
  - Lattice Wigner transform
  - Time-dependent perturbation expansion
  - Expectation values as graphs
  - Classification of graphs

- References

Classical harmonic crystal

We study a *cubic* lattice, $y \in \mathbb{Z}^3$, with *elastic couplings* $\alpha(y)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} q_y(t) = v_y(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} v_y(t) = -\sum_{y' \in \mathbb{Z}^3} \alpha(y - y') q_{y'}(t)$$

- $q(0)$ and $v(0) \in \ell_2(\mathbb{Z}^3, \mathbb{R})$
- α is *non-zero*, *symmetric* $(\alpha(-y) = \alpha(y))$, and *exponentially decaying*
- Mechanical stability requires $\widehat{\alpha}(k) \geq 0$, we assume $\widehat{\alpha}(k) > 0$.

Time-evolution can be solved using Fourier-transform in terms of $\omega : \mathbb{T}^3 \to \mathbb{R}$, where $\mathbb{T}^3 =$ (unit 3-torus),

$$\omega(k) = \sqrt{\widehat{\alpha}(k)} \qquad \text{(dispersion relation)}$$

For $\sigma \in \{\pm 1\}$, $k \in \mathbb{T}^3$, and a time t, define

$$\widehat{\psi}_\sigma(k,t) = \frac{1}{2}\big(\omega(k)\widehat{q}(k,t) + \mathrm{i}\sigma\widehat{v}(k,t)\big).$$

Then for all $t \in \mathbb{R}$, $y \in \mathbb{Z}^3$,

$$\psi_\sigma(y,t) = \int_{\mathbb{T}^3} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}(2\pi k\cdot y - \sigma\omega(k)t)}\widehat{\psi}_\sigma(k,0)$$

$$\Rightarrow \quad q(t) = \Omega^{-1}(2\,\mathrm{Re}\,\psi_+(t)), \quad v(t) = 2\,\mathrm{Im}\,\psi_+(t); \quad \Omega = \mathcal{F}^{-1}\omega\mathcal{F}.$$

Time-evolution conserves the *total energy*: $E_0(q(t),v(t)) = E_0(q(0),v(0))$ for

$$E_0(q,v) = \frac{1}{2}\Big(\sum_{y\in\mathbb{Z}^3} v_y^2 + \sum_{y',y\in\mathbb{Z}^3} \alpha(y'-y)q_{y'}q_y\Big)$$

- $E_0(q,v) \geq 0$ and $E_0(q(t),v(t)) = \|\psi(t)\|^2$.
- $E_0(q,v) < \infty \quad \Leftrightarrow \quad q,v \in \ell_2$.

$W^\varepsilon[\phi]$ = *Wigner transform* of a vector $\phi \in \ell_2(\mathbb{Z}^3, \mathbb{C}^2)$: for $x \in \mathbb{R}^3$, $k \in \mathbb{T}^3$,

$$W^\varepsilon[\phi](x,k) = \int_{\mathbb{R}^3} \mathrm{d}p \; \mathrm{e}^{\mathrm{i}2\pi x\cdot p} \widehat{\phi}(k - \frac{1}{2}\varepsilon p) \otimes \widehat{\phi}(k + \frac{1}{2}\varepsilon p)$$
$$= \sum_{y',y\in\mathbb{Z}^3} \delta\big(x - \varepsilon\frac{y'+y}{2}\big)\mathrm{e}^{\mathrm{i}2\pi k\cdot(y'-y)} \phi_{y'} \otimes \phi_y$$

where $(a \otimes b)_{\sigma'\sigma} = a^*_{\sigma'} b_\sigma$ and $\varepsilon > 0$ sets the *spatial scale* to ε^{-1}.

- Determines the *energy density* $\mathcal{E}^\varepsilon$ by

$$\mathcal{E}^\varepsilon(x,t) = \int_{\mathbb{T}^3} \mathrm{d}k \, \mathrm{Tr}\, W^\varepsilon[\psi(t)](x,k) = 2\int_{\mathbb{T}^3} \mathrm{d}k \, W^\varepsilon_{++}[\psi(t)](x,k).$$

- $W^\varepsilon_{++}[\psi(t/\varepsilon)] \overset{*}{\rightharpoonup} W(t)$ when $\varepsilon \to 0^+$. $W(x,k,0) = \delta(x)|\widehat{\psi}_+(k)|^2$, and

$$\partial_t W(x,k,t) + \frac{\nabla\omega(k)}{2\pi} \cdot \nabla_x W(x,k,t) = 0, \qquad \text{for } t \in \mathbb{R}.$$

The energy density on a harmonic crystal is transported ballistically according to the group velocity of the dispersion relation ω.

. . . with small random mass perturbations

$$\frac{\mathrm{d}}{\mathrm{d}t} q_y(t) = v_y(t)$$

$$(1+\sqrt{\varepsilon}\,\xi_y)^{-2}\frac{\mathrm{d}}{\mathrm{d}t} v_y(t) = -\sum_{y'\in\mathbb{Z}^3} \alpha(y-y')q_{y'}(t)$$

$m_y = \frac{1}{(1+\sqrt{\varepsilon}\,\xi_y)^2}$, (ξ_y) i.i.d. with $\sup|\xi_0| < \infty$, $\mathbb{E}[\xi_0] = 0$, $\mathbb{E}[\xi_0^2] = 1$

$$\psi_\sigma(y,t) = \frac{1}{2}\left[(\Omega q(t))_y + \mathrm{i}\sigma(1+\sqrt{\varepsilon}\,\xi_y)^{-1} v(t)_y\right] \quad \Longrightarrow$$

$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = -\mathrm{i}H_\varepsilon\psi(t)$, with $H_\varepsilon = H_0 + \sqrt{\varepsilon}V$,

$$H_0 = \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} \Omega\xi + \xi\Omega & -\Omega\xi + \xi\Omega \\ \Omega\xi - \xi\Omega & -\Omega\xi - \xi\Omega \end{pmatrix}$$

Energy density

Again, the time-evolution conserves the total energy,

$$E(q,v) = \frac{1}{2}\Big(\sum_{y\in\mathbb{Z}^3} (1+\sqrt{\varepsilon}\,\xi_y)^{-2} v_y^2 + \sum_{y',y\in\mathbb{Z}^3} \alpha(y'-y) q_{y'} q_y \Big).$$

- $E(q(t),v(t)) = \|\psi(t)\|^2 = 2\|\psi_+(t)\|^2$, since $\psi_+^* = \psi_-$.
- $\mathcal{E}^\varepsilon(x,t) = \sum_{y\in\mathbb{Z}^3} \delta(x-\varepsilon y) 2|\psi_+(y;t)|^2 = 2\int_{\mathbb{T}^3} \mathrm{d}k\, W^\varepsilon_{++}[\psi(t)](x,k)$

We will study the weak-$*$ limit of the distribution $\mathbb{E}[W^\varepsilon_{++}[\psi(t/\varepsilon)]]$ for a fixed $t \geq 0$ when $\varepsilon \to 0^+$ (a kinetic limit).

Explicitly, given $t \geq 0$ and $J \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{T}^3)$, we will solve the limit of

$$\mathbb{E}[\langle J, W^\varepsilon_{++}[\psi(t/\varepsilon)]\rangle], \qquad \text{when } \varepsilon \to 0 \text{ via a positive sequence.}$$

Assumptions (initial conditions)

For every ε, there is given $\psi^\varepsilon \in \mathcal{H} = \ell_2(\mathbb{Z}^3, \mathbb{C}^2)$, independent of ξ, such that

(IC1) *Energy remains uniformly bounded:*

$$\sup_\varepsilon \|\psi^\varepsilon\|^2 < \infty.$$

(IC2) *The sequence is tight on the kinetic scale:*

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \sum_{|y|>R/\varepsilon} |\psi^\varepsilon_y|^2 = 0.$$

(IC3) *The weak-$*$ limit of $W^\varepsilon_{++}[\psi^\varepsilon]$ exists.*

Then we define $q(0) = \Omega^{-1}(2\,\mathrm{Re}\,\psi^\varepsilon_+)$, $v(0) = 2\,\mathrm{Im}\,\psi^\varepsilon_+$. Examples:

- ε-independent $\psi \in \mathcal{H}$.
- WKB-type $\psi^\varepsilon \in \mathcal{H}$: for $h, S \in \mathcal{S}(\mathbb{R}^3)$, S real, let

$$\psi^\varepsilon_{+,y} = \varepsilon^{3/2} h(\varepsilon y) \mathrm{e}^{\mathrm{i}S(\varepsilon y)/\varepsilon} \quad \text{and} \quad \psi^\varepsilon_- = (\psi^\varepsilon_+)^*.$$

Assumptions (dispersion relation)

The assumptions made for α imply:

(DR1) *ω is smooth and $\omega(-k) = \omega(k)$.*

(DR2) $\min_k \omega(k) > 0$ *and* $\max_k \omega(k) < \infty$

In addition, we need to assume:

(DR3, dispersivity) *For any smooth f, when $t \to \infty$,*

$$\left| \int_{\mathbb{T}^3} \mathrm{d}k\, f(k) \mathrm{e}^{-\mathrm{i}t\omega(k)} \right| = \mathcal{O}(t^{-3/2}).$$

True, if ω is a Morse function (only isolated critical points, all non-degenerate).

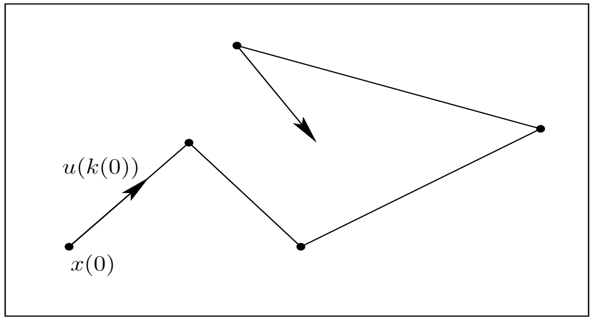
(DR4, suppression of crossings) $\exists\, 0 < \gamma \leq 1, n \in \mathbb{N}$, *such that when $\beta \to 0^+$,*

$$\sup_{u \in \mathbb{T}^3, \alpha \in \mathbb{R}^3} \int_{(\mathbb{T}^3)^2} \mathrm{d}k_1 \mathrm{d}k_2 \prod_{i=1}^{3} \frac{1}{|\alpha_i - \omega(k_i) + \mathrm{i}\beta|} = \mathcal{O}(\beta^{\gamma-1} (\ln \beta)^n)$$

where $k_3 = k_1 - k_2 + u$.

Expected to be true quite generally, but has only been proven for a few simplest cases. Related to the curvature of the surfaces $\{k : \omega(k) = \text{constant}\}$.

The Boltzmann equation



Consider the stochastic process $(x(t), k(t))$:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = u(k(t)); \quad u(k) = \frac{\nabla\omega(k)}{2\pi}.$$

$k(t) =$ jump process with av. waiting time $\dfrac{1}{\sigma_{\mathrm{tot}}(k)}$ and $\mathbb{P}(k \mapsto \mathrm{d}k') = \dfrac{\nu_k(\mathrm{d}k')}{\sigma_{\mathrm{tot}}(k)}$,

$$\nu_k(\mathrm{d}k') = \mathrm{d}k'\,\delta(\omega(k) - \omega(k'))\,2\pi\omega(k')^2, \qquad \sigma_{\mathrm{tot}}(k) = \int_{\mathbb{T}^3} \nu_k(\mathrm{d}k').$$

- Dispersivity (DR3) guarantees the process is well-defined and $\sigma_{\mathrm{tot}} \in C(\mathbb{T}^3)$.
- The associated probability density W satisfies a *linear Boltzmann equation*:

$$\partial_t W(x,k,t) + \frac{\nabla\omega(k)}{2\pi}\cdot\nabla_x W(x,k,t) = \int_{\mathbb{T}^3} \nu_k(\mathrm{d}k')\left(W(x,k',t) - W(x,k,t)\right)$$

Main theorem

Suppose $\varepsilon \longrightarrow 0$ via a positive sequence. Let ω satisfy (DR1) – (DR4), the initial conditions (ψ^ε) satisfy (IC1) – (IC3) and let $\psi(t)$ denote the random vector

$$\psi(t) = \mathrm{e}^{-\mathrm{i}tH_\varepsilon}\psi^\varepsilon.$$

Then for all $t \geq 0$, $J \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{T}^3)$, there exists a bounded positive Borel measure μ_t such that

$$\lim_{\varepsilon \longrightarrow 0} \mathbb{E}[\langle J, W^\varepsilon_{++}[\psi(t/\varepsilon)]\rangle] = \int_{\mathbb{R}^3 \times \mathbb{T}^3} \mu_t(\mathrm{d}x\,\mathrm{d}k)\, J(x,k)^*.$$

The measures μ_t coincide with the unique solution to the Boltzmann equation with initial conditions determined by μ_0.

Corollary: The time-evolution of the energy density in the kinetic limit can be solved by first solving the Boltzmann equation and then integrating out the k-variable.

Outline of the proof

1. Prove that it is enough to show that the Fourier-transform of the Wigner transform converges to the characteristic function of μ_t.
2. $H_\varepsilon = H_0 + \sqrt{\varepsilon} V$, iterate the Duhamel formula

$$\mathrm{e}^{-\mathrm{i}tH_\varepsilon} = \mathrm{e}^{-\mathrm{i}tH_0} + \int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(t-s)H_\varepsilon}(-\mathrm{i}\sqrt{\varepsilon}V)\mathrm{e}^{-\mathrm{i}sH_0}$$

   up to $\mathcal{O}(\ln \varepsilon)$ times $\implies$ time-dependent perturbation expansion of $\psi(t) = \psi_{\text{main}}(t) + \psi_{\text{err}}(t)$, with only the remainder $\psi_{\text{err}}(t)$ containing $\mathrm{e}^{-\mathrm{i}tH_\varepsilon}$.
3. Use unitarity (and partial time-integration) to show that $\mathbb{E}\left[\|\psi_{\text{err}}(t/\varepsilon)\|^2\right] \to 0$.
4. Thus it is enough to study the Fourier-transform of $W^\varepsilon_{++}[\psi_{\text{main}}(t/\varepsilon)]$, when $\mathbb{E}$ yields a sum over momentum graphs containing only the free propagator.
5. The sum over “ladder” graphs converges to the characteristic function of μ_t, while the rest yields a contribution vanishing as $\varepsilon \to 0$.

Properties of the lattice Wigner transform

- $\langle J, W^\varepsilon[\psi]\rangle = \langle \psi\,|\mathcal{W}^\varepsilon[J]\psi\rangle$, with $\|\mathcal{W}^\varepsilon[J]\| \le c\|J\|_{0,4}$.
- The *Fourier-transform of $W^\varepsilon[\psi]$* is given by a function on $\mathbb{R}^3 \times \mathbb{Z}^3$, $(p, n) \mapsto$

$$\sum_{y\in\mathbb{Z}^3} \psi^*_{y-n}\psi_y \mathrm{e}^{-\mathrm{i}2\pi\varepsilon p\cdot(y-\frac{1}{2}n)} = \int_{\mathbb{T}^3} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}2\pi n\cdot k}\widehat{\psi}(k - \frac{1}{2}\varepsilon p)^*\widehat{\psi}(k + \frac{1}{2}\varepsilon p).$$

- Any *weak-$*$ limit point of $W^\varepsilon[\psi^\varepsilon]$*, for $\|\psi^\varepsilon\|$ bounded, is a *bounded positive Borel measure* on $\mathbb{R}^3 \times \mathbb{T}^3$. If the sequence is tight, the Fourier-transforms of $W^\varepsilon[\psi^\varepsilon]$ converge to the characteristic function of this measure.
- If the Fourier transforms of $W^\varepsilon[\psi^\varepsilon]$ have a pointwise limit almost everywhere, then the sequence converges in the weak-$*$ topology (to the corr. measure).
- These results hold also for expectation values over a (ε-dependently) random ψ, if $\sup_\varepsilon \mathbb{E}[\|\psi\|^2] < \infty$.

Duhamel expansion

$$\mathrm{e}^{-\mathrm{i}tH_\varepsilon} = \sum_{N=0}^{N_0-1} F_N(t) + \sum_{N'=0}^{N_0'-1} \kappa \int_0^\infty \mathrm{d}r\, \mathrm{e}^{-\kappa(r-\underline{r})} \mathrm{e}^{-\mathrm{i}(t-\underline{r})H_\varepsilon} G_{N',N_0}(\underline{r};\kappa)$$

$$+ \int_0^t \mathrm{d}r\, \mathrm{e}^{-\mathrm{i}(t-r)H_\varepsilon} A_{N_0',N_0}(r;\kappa), \quad \text{with } \underline{r} = \min(t,r)$$

where, with $W_s = (-\mathrm{i}\sqrt{\varepsilon}V)\mathrm{e}^{-\mathrm{i}sH_0}$,

$$F_N(t) = \int_{\mathbb{R}_+^{N+1}} \mathrm{d}s\, \delta\Big(t - \sum_{\ell=1}^{N+1} s_\ell\Big) \mathrm{e}^{-\mathrm{i}s_{N+1}H_0} W_{s_N} \cdots W_{s_1}$$

$$G_{N',N}(t;\kappa) = \int_{\mathbb{R}_+^{N'+N+1}} \mathrm{d}s\, \delta\Big(t - \sum_{\ell=1}^{N'+N+1} s_\ell\Big) \mathrm{e}^{-\kappa \sum_{j=N+1}^{N+N'} s_j} \mathrm{e}^{-\mathrm{i}s_{N+N'+1}H_0} \prod_{j=1}^{N+N'} W_{s_j}$$

$$A_{N',N}(t;\kappa) = \int_{\mathbb{R}_+^{N'+N}} \mathrm{d}s\, \delta\Big(t - \sum_{\ell=1}^{N'+N} s_\ell\Big) \mathrm{e}^{-\kappa \sum_{j=N+1}^{N+N'} s_j} \prod_{j=1}^{N+N'} W_{s_j}$$

Choice of cut-off parameters

Notation: $\langle x \rangle = \sqrt{1+x^2}.$

Let γ be the constant in (DR4), and let

$$\gamma' = \min\left(\frac{1}{2}, \gamma\right), \quad a_0 = \frac{\gamma'}{40} \quad \text{and} \quad b_0 = 40\left(1 + \frac{2}{\gamma'}\right),$$

$$N_0(\varepsilon) = \max\left(1, \left\lfloor \frac{a_0\,|\ln\varepsilon|}{\ln\langle\ln\varepsilon\rangle} \right\rfloor\right), \quad N_0'(\varepsilon) = 8N_0(\varepsilon) \quad \text{and} \quad \kappa(\varepsilon) = \varepsilon\langle\ln\varepsilon\rangle^{b_0},$$

$$\psi_{\text{main}}(t) = \sum_{N=0}^{N_0(\varepsilon)-1} F_N(t)\psi^\varepsilon \quad \text{and} \quad \psi_{\text{err}}(t) = \mathrm{e}^{-itH_\varepsilon}\psi^\varepsilon - \psi_{\text{main}}(t).$$

Then, in the limit $\varepsilon \longrightarrow 0$ we have $N_0 \longrightarrow \infty$, $\kappa \longrightarrow 0$, and

$$c^N N!\langle\ln\varepsilon\rangle^{N+d}\varepsilon^{\gamma'} \longrightarrow 0, \quad \text{and} \quad \varepsilon^{-2}\left(\frac{\varepsilon}{\kappa}\right)^{N_0} c^N N!\langle\ln\varepsilon\rangle^{N+d} \longrightarrow 0,$$

whenever $N = rN_0(\varepsilon)$, with $0 \le r < 20$ and $c, d \ge 0$ being arbitrary constants.

Analysis of the error term

$$\mathbb{E}\big[\|\psi_{\text{err}}(t/\varepsilon)\|^2\big] \le 2t^2\varepsilon^{-2} \sup_{0\le r\le t/\varepsilon} \mathbb{E}\big[\|A_{N_0',N_0}(r;\kappa)\psi^\varepsilon\|^2\big]$$

$$+\,2(t\kappa/\varepsilon+1)^2(N_0')^2 \sup_{\substack{0\le N'\le N_0'-1\\ 0\le r\le t/\varepsilon}} \mathbb{E}\big[\|G_{N',N_0}(r;\kappa)\psi^\varepsilon\|^2\big]$$

There are constants c and c' and ε_1, such that for $0 \le t' \le t/\varepsilon$, $0 \le N' < N_0'$,

$$\mathbb{E}\big[\|G_{N',N_0}(t';\kappa)\psi^\varepsilon\|^2\big] \le c'\|\psi^\varepsilon\|^2(cT)^{\frac{\bar N}{2}}$$

$$\times\left[\bar N!\left\langle \ln\frac{T}{\varepsilon}\right\rangle^{\bar N+\max(2,d_2)} \bar N^{\max(1,d_1)}\frac{T}{\langle t/\varepsilon\rangle^{\gamma'}} + \frac{1}{\lfloor N_0/2\rfloor!}\right],$$

$$\mathbb{E}\big[\|A_{N_0',N_0}(t';\kappa)\psi^\varepsilon\|^2\big] \le c'\|\psi^\varepsilon\|^2\,(cT)^{\frac{\bar N}{2}}\,\bar N!\,\left\langle \ln\frac{T}{\varepsilon}\right\rangle^{\bar N}\left[\varepsilon^3 + \left(\frac{\varepsilon}{\kappa}\right)^{N_0}\right]$$

where $T = \langle t\rangle$ and $\bar N = 2(N_0 + N_0') = 18N_0$. Therefore,

$$\lim_{\varepsilon\to 0} \mathbb{E}\big[\|\psi_{\text{err}}(t/\varepsilon)\|^2\big] = 0$$

Expectation values as graphs

$$\text{“} \sum_{y\in\mathbb{Z}^3} \mathrm{e}^{-\mathrm{i}2\pi y\cdot k} = \delta(k) \text{ ”},$$

$$\mathbb{E}\Big[\prod_{\ell=1}^{N} \xi_{i_\ell}\Big] = \sum_{S\in\pi(I_N)} \prod_{A\in S} \Big[C_{|A|} \sum_{y\in\mathbb{Z}^3} \prod_{\ell\in A} \delta_{i_\ell,y}\Big], \quad C_n = n\text{:th cumulant}$$

$$\int_{\mathbb{R}_+^N} \mathrm{d}s\, \delta\Big(t - \sum_{\ell=1}^{N} s_\ell\Big) \prod_{\ell=1}^{N} \mathrm{e}^{-\mathrm{i}s_\ell w_\ell} = -\oint_\Gamma \frac{\mathrm{d}z}{2\pi} \mathrm{e}^{-\mathrm{i}tz} \prod_{\ell=1}^{N} \frac{\mathrm{i}}{z - w_\ell}$$

$\Longrightarrow$ The expectation value is a sum over all *partitions* S with an *amplitude*,

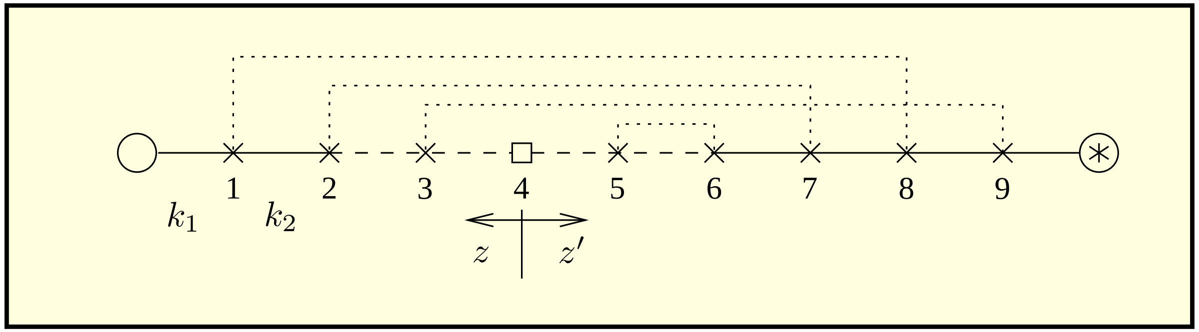
$$(-\mathrm{i})^{\bar{N}_1-\bar{N}_2} \varepsilon^{\bar{N}/2} \int_{\mathbb{T}^3} \mathrm{d}\eta_0 \oint_{\Gamma_\beta} \frac{\mathrm{d}z}{2\pi} \oint_{\Gamma_\beta} \frac{\mathrm{d}z'}{2\pi} \mathrm{e}^{-\mathrm{i}t(z+z')} \int_{(\mathbb{T}^3)^{\bar{N}+1}} \mathrm{d}\eta\, \delta(\eta_{\bar{N}_1+1} + p)$$

$$\times \prod_{A\in S} \Big(C_{|A|} \delta\Big(\sum_{\ell\in A} \eta_\ell\Big)\Big) \widehat{\psi}^{\varepsilon}(\eta_0 - p)^{\dagger} \Big[\prod_{\ell=\bar{N}_1+3+N'}^{\bar{N}+2} \Big(\frac{\mathrm{i}}{z' + H(k_\ell)} v(k_\ell, k_{\ell-1})\Big)$$

$$\times \prod_{\ell=\bar{N}_1+3}^{\bar{N}_1+2+N'} \Big(\frac{\mathrm{i}}{z' + \mathrm{i}\kappa + H(k_\ell)} v(k_\ell, k_{\ell-1})\Big) \frac{\mathrm{i}}{z' + \mathrm{i}\kappa + H(k_{\bar{N}_1+2})} f\Big(k_{\bar{N}_1+1} - \frac{1}{2}p\Big) \frac{\mathrm{i}}{z + \mathrm{i}\kappa - H(k_{\bar{N}_1+1})}$$

$$\times \prod_{\ell=N_1+1}^{N_1+N'} \Big(v(k_{\ell+1}, k_\ell) \frac{\mathrm{i}}{z + \mathrm{i}\kappa - H(k_\ell)}\Big) \prod_{\ell=1}^{N_1} \Big(v(k_{\ell+1}, k_\ell) \frac{\mathrm{i}}{z - H(k_\ell)}\Big) \widehat{\psi}^{\varepsilon}(\eta_0)\Big]; \quad \eta_\ell = k_{\ell+1} - k_\ell$$

An example with $N_1 = 3$, $N_2 = 5$, $N' = 2$, and a partition S with $|S| = 4$:



- ○ The initial state.
- · · · Dotted line connects a cluster $A \in S$, each yielding $C_{|A|}\delta(\sum_{\ell\in A}(k_{\ell+1} - k_\ell))$.
- —— Solid line is a *propagator*, a matrix factor $\frac{\mathrm{i}}{z-H(k)}$, $H(k) = \mathrm{diag}(\omega(k), -\omega(k))$.
- - - - A dashed line is a restricted propagator, with $z + \mathrm{i}\kappa$ instead of z above.
- × A cross denotes an *interaction vertex*, a matrix factor $\sqrt{\varepsilon}\, v(k', k)$.
- □ The box denotes the *observable*, $\mathbb{1}\delta(k_{N_1+2} - k_{N_1+1})$ for norm, $P_{++}\mathrm{e}^{\mathrm{i}2\pi n\cdot(k_{N_1+2}-k_{N_1+1})}\delta(k_{N_1+2} - k_{N_1+1} - \varepsilon p)$ otherwise.

After the observable, the basic propagators are $\frac{\mathrm{i}}{z'+H(k)}$, instead of $\frac{\mathrm{i}}{z-H(k)}$.

Classification of graphs

A graph corresponding to a partition S of $N = N_1 + N_2$ vertices is

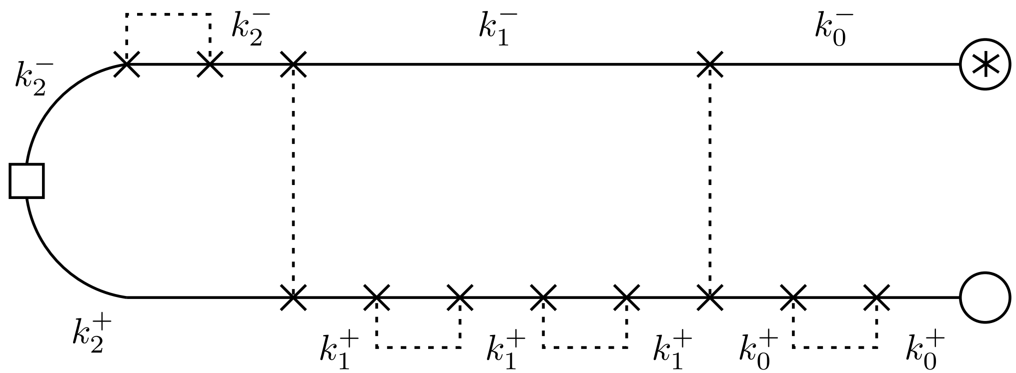
irrelevant, if S contains a singlet. Then the amplitude is zero, as $C_1 = \mathbb{E}[\xi_0] = 0$.

higher order, if there is $A \in S$ such that $|A| > 2$. Each such cluster yields an additional factor $\varepsilon^{(|A|-2)/2}$, and their sum is $\mathcal{O}(\varepsilon^{1/2})$.

crossing, if it is a pairing which contains two pairs crossing each other: $\exists\{i_1, i_2\}$, $\{j_1, j_2\} \in S$ such that $i_1 < j_1 < i_2 < j_2$. The amplitude is $\mathcal{O}(\varepsilon^{\gamma})$.

nested, if it is none of the above, but there is a pairing $\{i_1, i_2\} \in S$ such that $i_1 + 1 < i_2$ and either $i_1 \geq N_1 + 2$ or $i_2 \leq N_1$. The amplitude is $\mathcal{O}(\varepsilon^{1/2})$.

simple, otherwise (amplitude $\approx \mathcal{O}(1/N!)$). An example:



References

This talk: J. Lukkarinen, and H. Spohn, *arxiv.org* preprint: *math-ph / 0505075*

From Random Schrödinger to Boltzmann equation

H. Spohn, J. Stat. Phys. **17** (1977) 385–412

L. Erdős, and H.-T. Yau, Commun. Pure Appl. Math. **53** (2000) 667–735

T. Chen, *arxiv.org* preprints: *math-ph / 0305051* and *math-ph / 0407037*

G. Bal, T. Komorowski, and L. Ryzhik, Commun. Math. Phys. **242** (2003) 81–135

Rigorous diffusion limits

L. Erdős, M. Salmhofer, and H.-T. Yau, *arxiv.org* preprint: *math-ph / 0502025*

T. Komorowski, and L. Ryzhik, *arxiv.org* preprint: *math-ph / 0505082*

On general phonon Boltzmann equations

H. Spohn, *arxiv.org* preprint: *math-ph / 0505025*

General references to Wigner transforms

L. Ryzhik, G. Papanicolaou, and J. B. Keller, Wave Motion **24** (1996) 327–370

P. Gérard, *et al.*, Commun. Pure Appl. Math. **50** (1997) 323–379

A. Mielke, WIAS preprint No. 1027 (2005)