Travelling breathers with exponentially small tails in nonlinear oscillator chains

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1 – Discrete travelling waves and breathers

We consider one-dimensional lattices (hamiltonian case)

$$\frac{d^2 u_n}{dt^2} + W'(u_n) = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}),\ n \in \mathbb{Z}$$

where

- u_n : displacement of the nth particle from an equilibrium position
- V, W : smooth interaction and on-site potentials, $V'(0) = W'(0) = 0$, $V''(0), W''(0) > 0$.

Fermi-Pasta-Ulam (FPU) lattice : $W = 0$

Klein-Gordon lattice : $V(x) = \frac{\gamma}{2}x^2$

Different solution classes :

- Travelling waves :

$$u_n(t) = u_{n-1}(t-\tau)$$

($\tau \neq 0$ is fixed), $u_n(t) = u_0(t - n\tau)$, velocity $1/\tau$.

- Pulsating travelling waves (commensurate case) :

$$u_n(t) = u_{n-p}(t - p\,\tau)$$

$\tau \neq 0$ and $p \in \mathbb{N}$ ($p \geq 2$) are fixed.

Equivalent definition : $u_n(t) = u(n - \frac{t}{\tau}, t)$, $u(x, \theta)$ is periodic in θ, period $\frac{p}{k}\tau$

- Static "breather" :

-Spatially localized : as $n \to \pm\infty$, $u_n(t) \to 0$
(or $u_n(t) \to an + c_\pm$ for $W = 0$).

-Time-periodic : $u_n(t + T_b) = u_n(t)$

- "Travelling breather" : pulsating travelling wave with spatial localization.

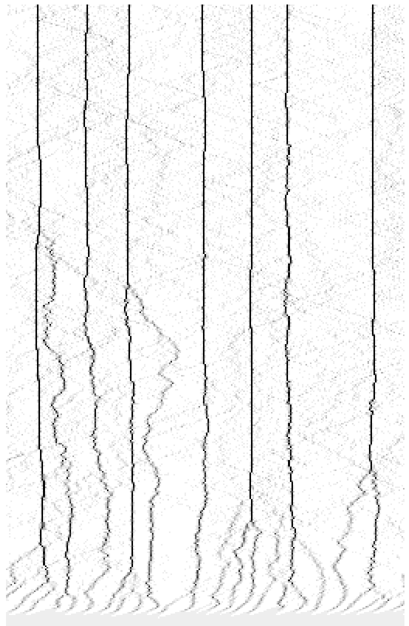


Fig. 2. Long term evolution of the energy density along the chain for an initial condition corresponding to a noisy wave given by Eq. (7). The horizontal axis indicates the position along the chain and the vertical axis corresponds to time, going upward. The energy density at site n is shown by a grey scale from $E_n = 0$ (white) to the maximum E_n recorded during the simulation (black).

FIG. 1: Modulational instability in a Klein-Gordon lattice with $W(x) = \frac{1}{2}(1 - e^{-x})^2$ (Morse potential). Space-time evolution (time is going upward) of the energy density (grey scales). Energy density goes from 0 (white) to a maximum value (black). The initial condition is a linear wave with a slightly modulated amplitude (small Gaussian noise). From I. Daumont, T. Dauxois and M. Peyrard, Nonlinearity 10 (1997).

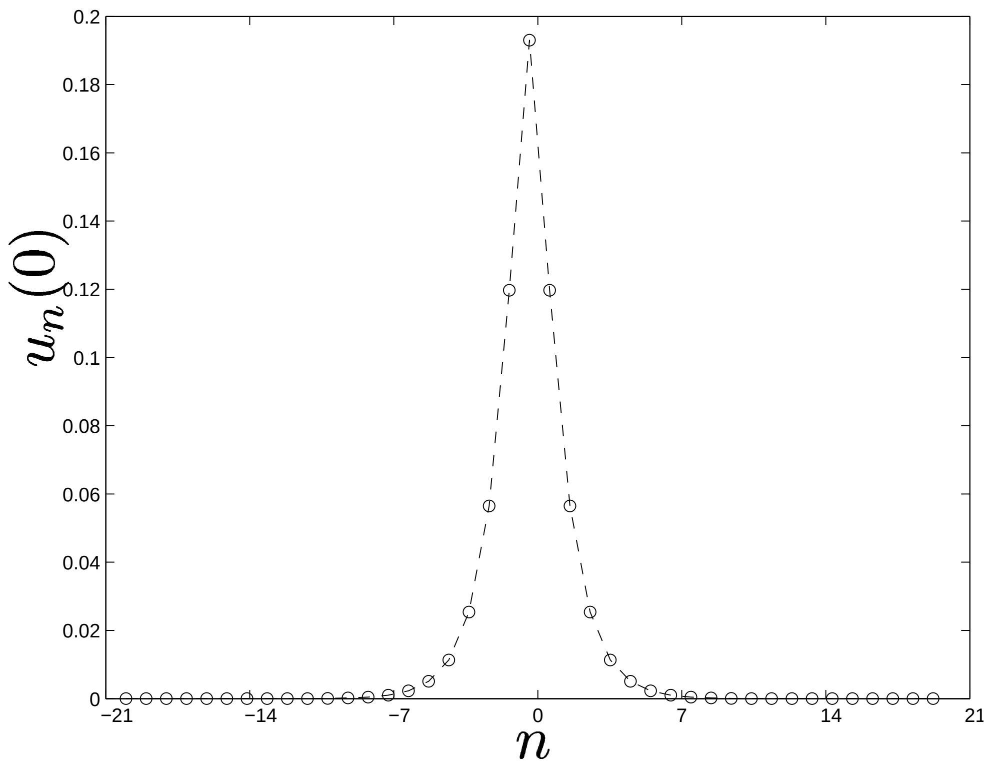


FIG. 2: Static breather solution for the Klein-Gordon lattice (periodic boundary conditions). On-site potential : $W(x) = \frac{1}{2}(1-e^{-x})^2$ (Morse potential). Breather period : $T_b = 6,32$. Coupling $\gamma = 0,03$.

Uncoupled or “anticontinuous” limit : MacKay and Aubry (1994)

Numerical application : Marin and Aubry (1996)

Diatomic FPU lattices with large mass ratio : Livi, Spicci, MacKay (1997).

Numerical computation of travelling breathers in Klein-Gordon lattices :

Case $p = 2$: $u_n(t) = u_{n-2}(t - T)$.

Y. Sire and G.J., Physica D 204 (2005).

$$\frac{d^2 u_n}{dt^2} + W'(u_n) = \gamma(u_{n+1} - 2u_n + u_{n-1}), \; n \in \mathbb{Z}$$

The potential W is even, $W''(0) = 1$

$$u_n(t) = (-1)^{n+1} u(\frac{t}{T} - \frac{(n-1)}{2}) \Rightarrow u_n(t) = u_{n-2}(t - T).$$

$$\frac{d^2 u}{dt^2} + T^2 \, W'(u) = -\gamma T^2 (u(t + 1/2) + 2u(t) + u(t - 1/2))$$

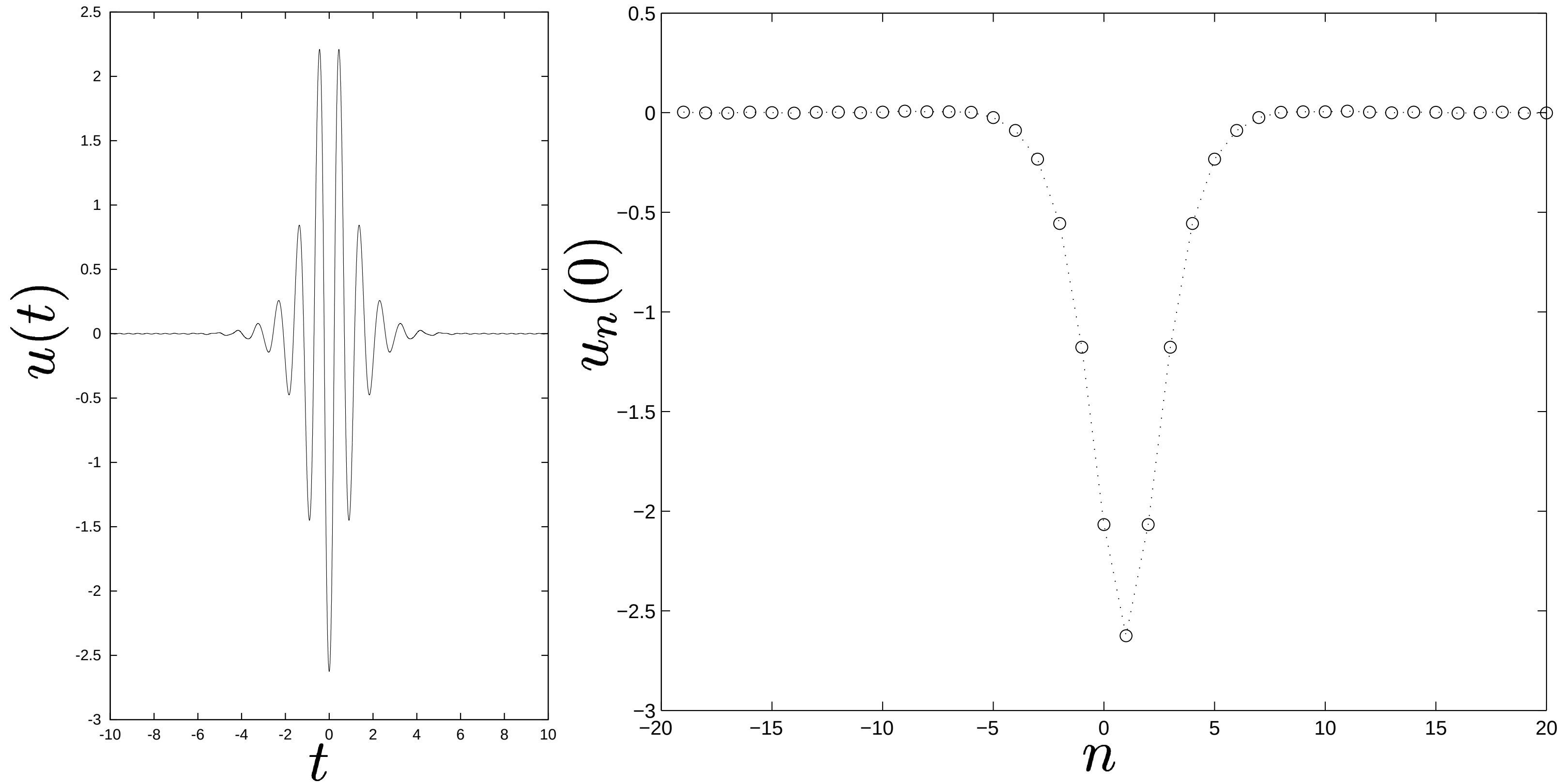


FIG. 3: Large amplitude solution for the trigonometric potential $W(x) = 1 - \cos(x)$ with $T = 8.1, \gamma \approx 0.9$. The left figure shows the displacements of mass $n = 1$ as a function of t (note that t is a rescaled time). The right one shows displacements for all lattice sites at time $t = 0$.

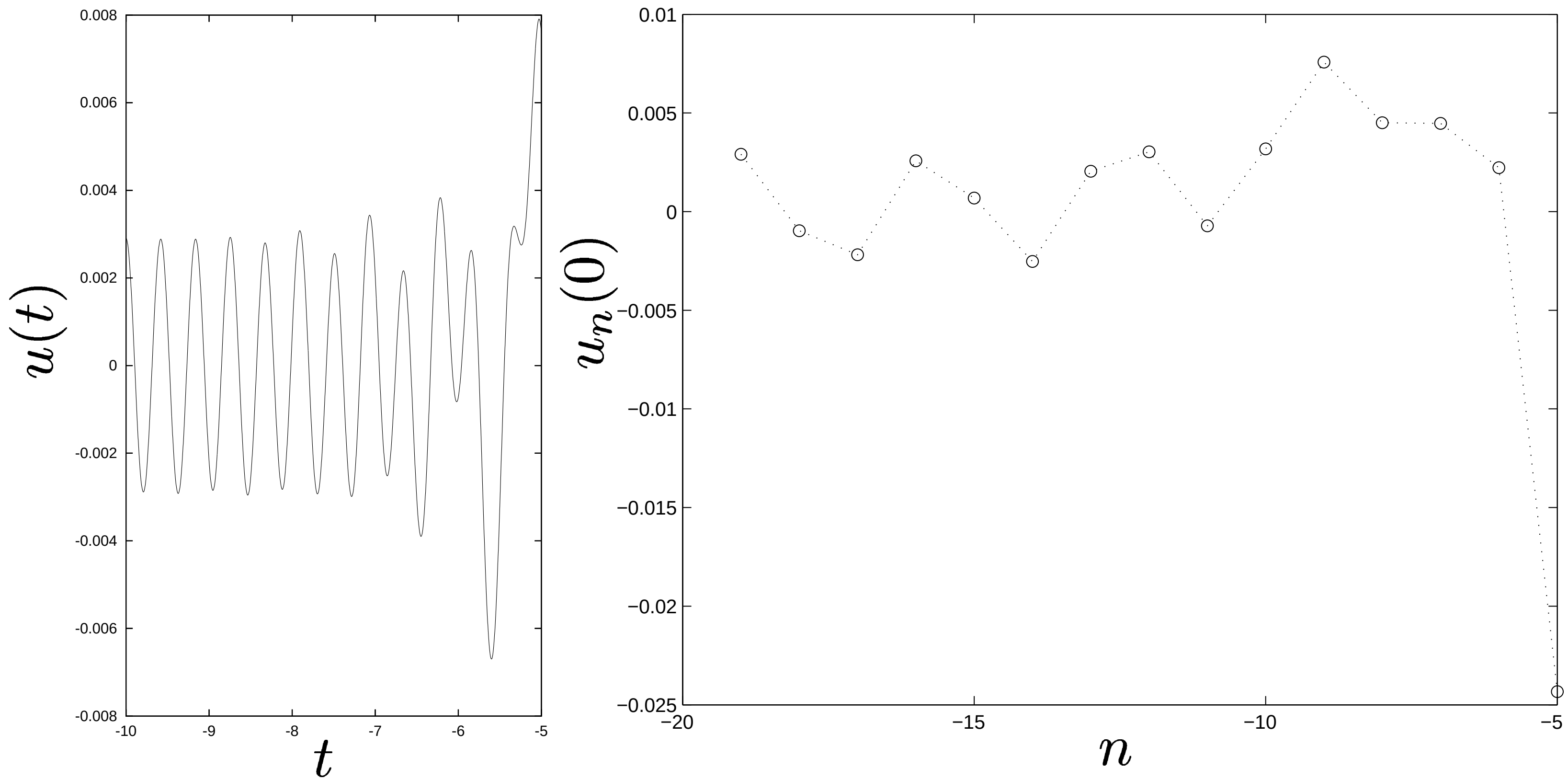


FIG. 4: Magnification of the solution tail ($W(x) = 1 - \cos(x), T = 8.1, \gamma \approx 0.9$).

Former detection of a "tail" for solitary wave solutions, i.e. $u_n(t) = u_{n-1}(t - T)$ ($p = 1$) : Aubry and Cretegny (1998)

2 – Spatial dynamics and center manifold reduction

- For travelling waves $u_n(t) = y(x)$, $x = n - t/\tau$ and

$$\frac{1}{\tau^2}\frac{d^2y}{dx^2} + W'(y) = V'(y(x+1) - y(x)) - V'(y(x) - y(x-1))$$

Scalar *advance-delay* or *mixed type* differential equation.

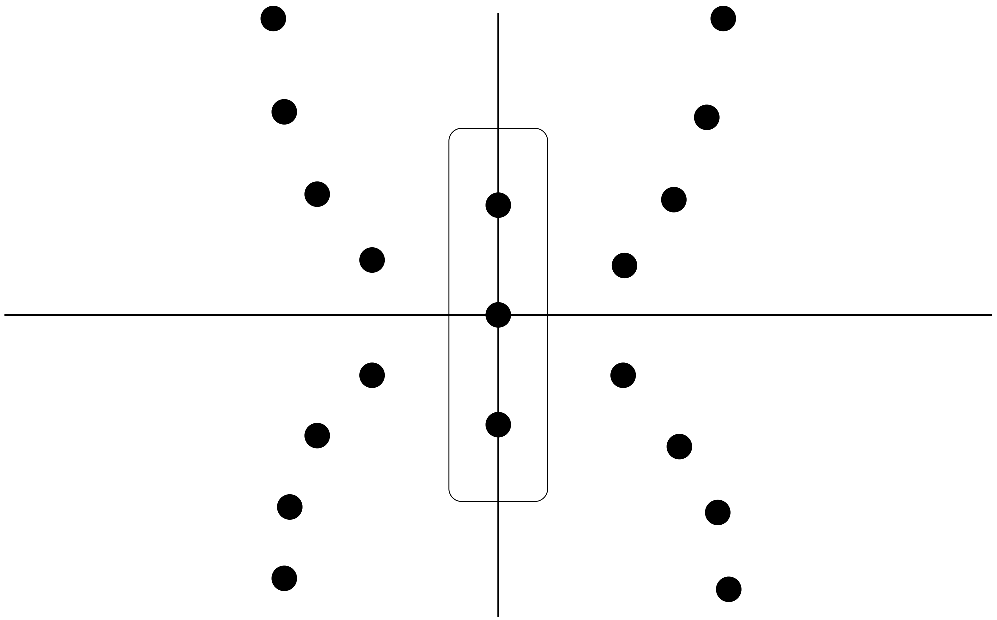
Evolution problem in the spatial coordinate x :
Iooss and Kirchgässner (2000), Iooss (2000).

Linear dispersion relation : $y(x) = y_0\, e^{z\,x}$ $\quad (W''(0) = \omega_0^2, V''(0) = \gamma)$

$$\frac{z^2}{\tau^2} + \omega_0^2 - 4\gamma \sinh^2\left(\frac{z}{2}\right) = 0$$

The spectrum is discrete (isolated eigenvalues with finite multiplicities) and
-unbounded, invariances $z \to -z$, $z \to \bar{z}$
-finite number of purely imaginary eigenvalues

Spatial dynamics and center manifold reduction



For small amplitude solutions in the nonlinear case :
center manifold reduction = "slaving principle"
modes that explode as $x \to \pm\infty$ are functions of "marginal" modes.

Spatial dynamics and center manifold reduction

➠ For time-periodic solutions (e.g. static breathers)

$$u_n(t) = y_n(\omega_b\, t), \quad y_n(t+2\pi) = y_n(t)$$

$$V'(y_{n+1} - y_n) = \omega_b^2 \frac{d^2 y_n}{dt^2} + W'(y_n) + V'(y_n - y_{n-1})$$

Since V' is locally invertible, the problem can be formulated as a map in a loop space :

$$(y_n, y_{n-1}) \to (y_{n+1}, y_n)$$

Evolution problem in the discrete spatial coordinate n

Discrete "slaving principle" for small amplitude solutions :
modes that explode as $n \to \pm\infty$ are functions of "marginal" modes (with respect to the dynamics in n !).

- Breathers in FPU lattices : G.J., C. R. Acad. Sci. Paris, t. 332, Série I (2001).
- Center manifold theorem for discrete systems with "spectral separation" : G.J., J. Nonlinear Sci. 13 (2003).
- Application to diatomic FPU lattices, arbitrary mass ratio : G.J. and P. Noble, Physica D 196 (2004).

➠ For pulsating travelling waves (including travelling breathers) :

$$u_n(t) = u_{n-p}(t - p\,\tau)$$

* Klein-Gordon lattice, with $p = 2$:

G.J. and Y. Sire, Commun. Math. Phys. 257 (2005).

* Fermi-Pasta-Ulam lattice, arbitrary p :

G. Iooss and G.J., Chaos 15 (2005).

* Klein-Gordon lattice, arbitrary p :

Y. Sire, *J. Dyn. Diff. Eq.*, to appear in 2005.

3 – Pulsating travelling waves in Fermi-Pasta-Ulam lattices

G. Iooss and G.J., Chaos 15 (2005)

$$\frac{d^2 u_n}{dt^2} = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}), \ n \in \mathbb{Z}$$

$$u_n(t) = u_{n-p}(t - p\,\tau)$$

We fix $p \geq 1$, $\tau > 0$, $V''(0) = 1$

Formulation in a moving frame : $u_n(t) = y_n(x)$, $x = n - t/\tau$

Equivalently : $y_n(x) = u_n(\tau(n - x))$

$$y_{n+p}(x) = y_n(x)$$

$$\frac{1}{\tau^2}\,\frac{d^2 y_n}{dx^2} = V'(y_{n+1}(x+1) - y_n(x)) - V'(y_n(x) - y_{n-1}(x-1)).$$

Formulation as a *reversible* evolution problem in a function space :

Additional variable : $Y_n(x, v) = y_n(x + v), \quad \xi_n = \frac{dy_n}{dx}$

$$U_n = (y_n, \xi_n, Y_n(v))^T, \quad U_{n+p} = U_n, \quad Y_{n|v=0} = y_n$$

For all $x \in \mathbb{R}$, $U(x) = (U_n(x))_{n\in\mathbb{Z}}$ is a p-periodic sequence in $\mathbb{R}^2 \times C^1(\,[-1, 1]\,)$

$$\frac{dU}{dx} = L_\tau U + \tau^2\, M(U)$$

$$(L_\tau U)_n = \begin{pmatrix} \xi_n \\ \tau^2(\delta_1 Y_{n+1} - 2y_n + \delta_{-1} Y_{n-1}) \\ \frac{dY_n}{dv} \end{pmatrix}$$

$M(U)$ belongs to the domain of L_τ, $M(U) = O(\,\|U\|^2\,)$ as $U \to 0$

Spectrum of $L_\mathcal{T}$ on the imaginary axis

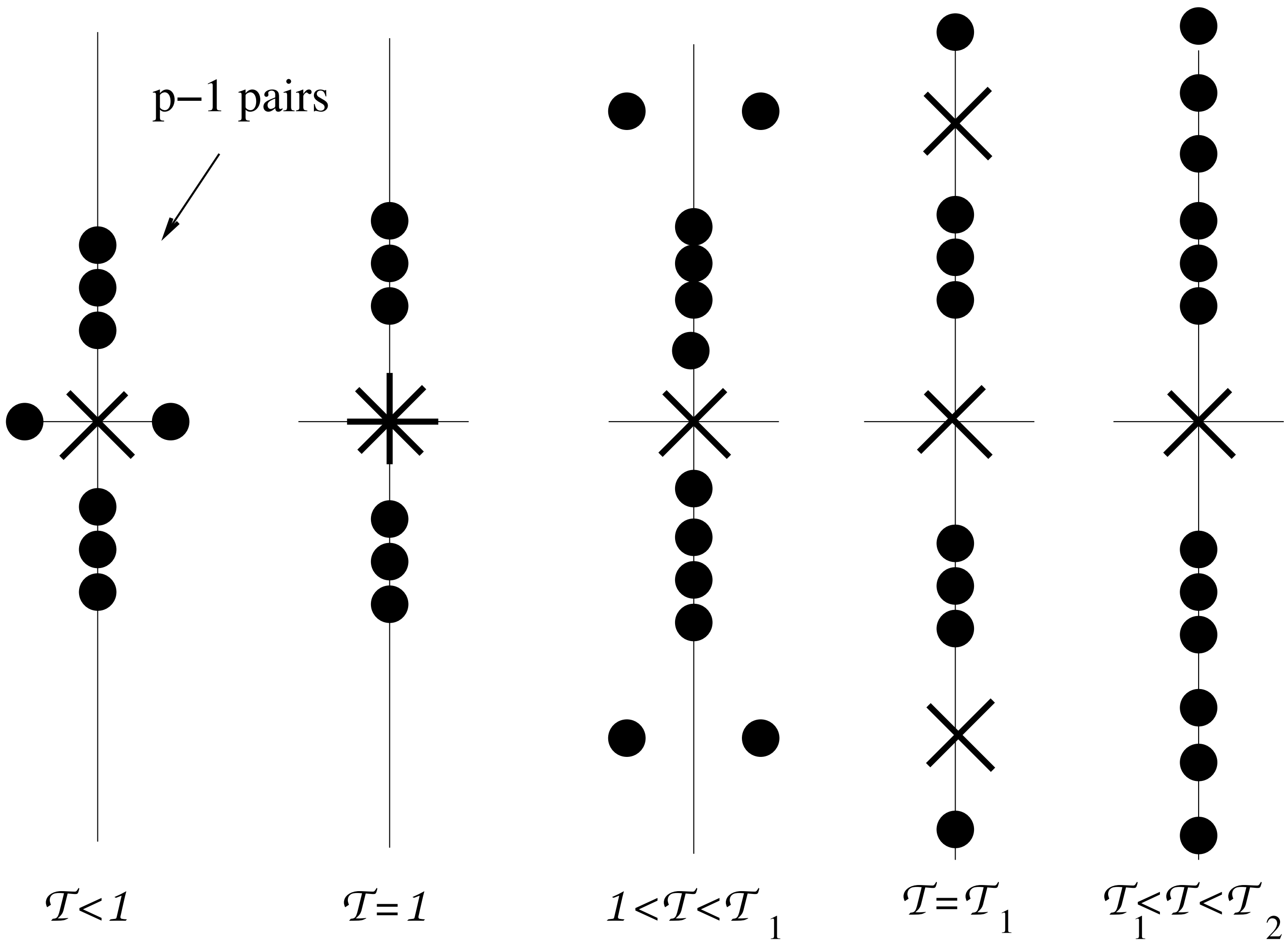


FIG. 5: Simple eigenvalue : •. Double eigenvalue : ×. Quadruple eigenvalue : ∗.

Double non semi-simple eigenvalues $i\lambda \neq 0$ occur (in pairs) for critical parameter values $\tau = \tau_k$ given by the solutions of :

$$\tau \Big| \cos\big(\frac{\lambda}{2} - \pi\frac{m}{p}\big)\Big| = 1, \quad \frac{\lambda}{2} = \tan\big(\frac{\lambda}{2} - \pi\frac{m}{p}\big),$$

$m \in \{\, 0, \ldots, p-1 \,\}, 1 < \tau_1 < \tau_2 < \ldots < \tau_k < \tau_{k+1} < \ldots$.

Case $\tau \approx \tau_k$: $2N + 6$ marginal modes

Spectrum of $L = L_{\tau_k}$ on the imaginary axis :

* $N = p + 2(k-1)$ pairs of simple eigenvalues $\pm i\lambda_j$ ($j = 1, \ldots, N$), λ_j being associated with $m = m_j$ and an eigenvector ζ_j,
* 2 pairs of double eigenvalues $\pm i\lambda_0$, λ_0 being associated with $m = m_0$, an eigenvector ζ_0 and a generalized eigenvector η_0,
* the double eigenvalue 0, associated with $m = 0$, an eigenvector χ_0 and a generalized eigenvector χ_1.

4 – Reduction and normal form

Reduction and normal form

Small amplitude solutions satisfy : $\operatorname{Sup}_{x\in\mathbb{R}} \; || \frac{dU}{dx} || \approx 0$

THEOREM 1 :

For $\tau \approx \tau_k$, small amplitude solutions have the form

$$\begin{aligned} U(x) &= A(x)\zeta_0 + B(x)\eta_0 + \sum_{j=1}^{N} C_j(x)\zeta_j + c.c. + D(x)\chi_1 + q(x)\chi_0 \\ &\quad + \psi(A(x), B(x), C(x), \bar{A}(x), \bar{B}(x), \bar{C}(x), D(x), \tau), \end{aligned}$$

where $C = (C_1, \ldots, C_N)$.

Coordinates of solutions on a $2N+6$-dimensional center manifold :

$$(A, B, C_1, \ldots, C_N, \bar{A}, \bar{B}, \bar{C}_1, \ldots, \bar{C}_N, D, q) \in \mathbb{C}^{2N+4} \times \mathbb{R}^2$$

$\psi \in C^m(\mathbb{C}^{2N+4} \times \mathbb{R}^2, \mathbb{D})$, $\mathbb{D}$ is the function space where $U(x)$ lies

$\psi(0, \tau) = 0, D\psi(0, \tau_k) = 0.$

Reduction and normal form

THEOREM 2 : for $\tau \approx \tau_k$, the problem is locally reduced to the *reversible* system

$$
\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D) + h.o.t.,\\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D) + h.o.t.,\\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j\mathcal{Q}_j(|A|^2, I, Q, D) + h.o.t., \quad (j = 1, \dots N)\\
\frac{dD}{dx} &= 0,\\
\frac{dq}{dx} &= D + \chi_1^*(\psi(A, B, C, \bar{A}, \bar{B}, \bar{C}, D, \tau)).
\end{aligned}
$$

$Q = (|C_1|^2, \dots, |C_N|^2)$, $I = i(A\bar{B} - \bar{A}B)$

$\mathcal{P}, \mathcal{S}, \mathcal{Q}_j$: polynomials with real coefficients (smooth in τ for $\tau \approx \tau_k$). χ_1^* : linear form.

The principal part of the system is a cubic polynomial in $A, B, C_1, \dots, C_N$, their conjugates, and D. Higher order terms do not depend on q.

5 – Bifurcating homoclinic solutions

Bifurcating homoclinic solutions

The *truncated* normal form is integrable

$$\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D), \\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D), \\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j\mathcal{Q}_j(|A|^2, I, Q, D), \quad (j = 1, \dots N) \\
\frac{dD}{dx} &= 0.
\end{aligned}$$

First integrals :

$$D, Q = (|C_1|^2, \dots, |C_N|^2), I = i(A\bar{B} - \bar{A}B), |B|^2 - \int_0^{|A|^2} \mathcal{S}(\alpha, I, Q, D)\, d\alpha.$$

All solutions deduced from reference : Iooss-Pérouème (1993), 1 :1 resonance with reversibility.

Bifurcating homoclinic solutions

Set $V(x) = \frac{1}{2}x^2 + \frac{\alpha}{3}x^3 + \frac{\beta}{4}x^4 +$ h.o.t (assume α or β nonzero).

Existence of homoclinic solutions to 0 in the *focusing case*

$$b - c_k^2(b + 2\alpha^2) > 0 \tag{1}$$

where $c_k = \frac{1}{\tau_k}$, $0 < c_k < 1$, c_k densely covers $[0, 1]$ for $p, k \geq 1$

Condition (1) can be also derived with multi-scale expansions : Tsurui (1972)

Small amplitude static FPU breathers exist (slightly above the phonon band) for $b > 0$; G.J. (2001)

$$b = 3\beta - 4\alpha^2$$

- For even V ($\alpha = 0$), homoclinic solutions exist for $\beta > 0$ (hard potential case)

  More complex situation if $\alpha \neq 0$:

- If $b < 0$, homoclinic orbits to 0 do not exist since $c_k < 1$ (e.g. $\beta \leq 0$).
- If $b > 0$, homoclinic orbits to 0 exist on the velocity interval $0 \leq c_k < c_{max}$,

$$c_{max}^2 = \frac{b}{b + 2\alpha^2} < 1.$$

Bifurcating homoclinic solutions

The solutions of the truncated normal form yield approximate (leading order) solutions of the FPU system

$$\begin{aligned} u_n(t) &\approx A(n-t/\tau)\, e^{-2i\pi m_0 n/p} + \sum_{j=1}^{N} \left(C_j(n-t/\tau)\, e^{-2i\pi m_j n/p} \right) \\ &\quad + c.c. + q(n-t/\tau), \end{aligned} \qquad (2)$$

with $\frac{dq}{dx} = D$.

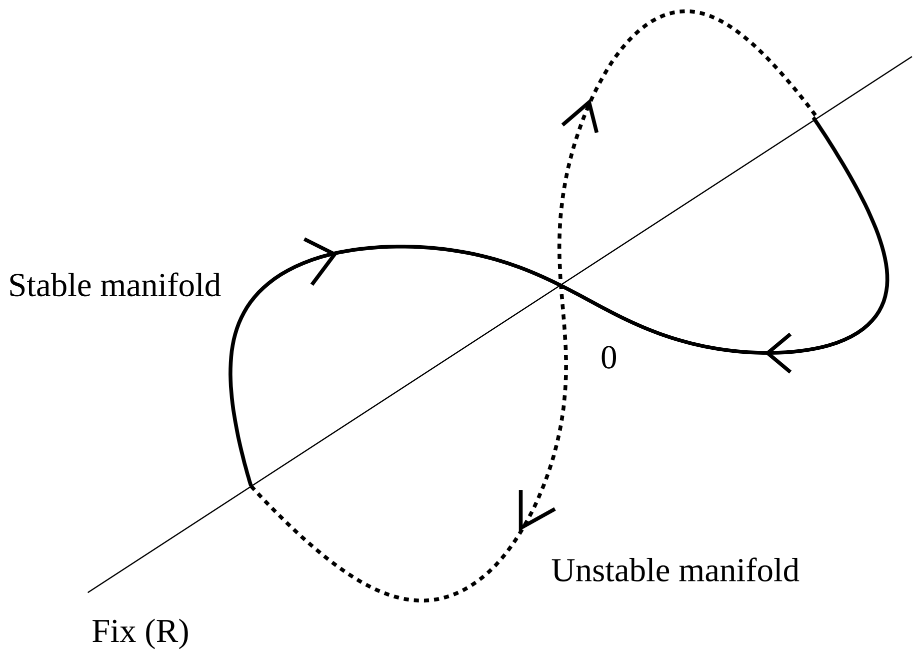
Do homoclinic solutions to 0 ($C_j = D = 0$) “persist” as higher order terms are taken into account in the normal form ?

i.e. do exact travelling breathers exist ?

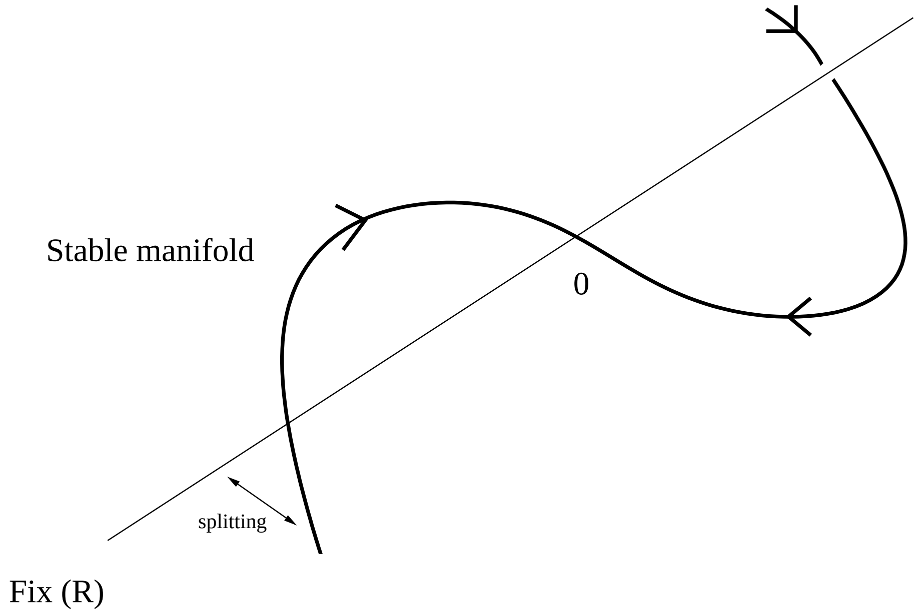
Case $D = 0$, normal form for $A, B, C_1, \ldots, C_N$.

Phase space : $2N + 4$-dimensional. Stable manifold of 0 : two-dimensional.

Reversibility symmetry R : dim Fix$(R) = N + 2$.



Case of the truncated normal form

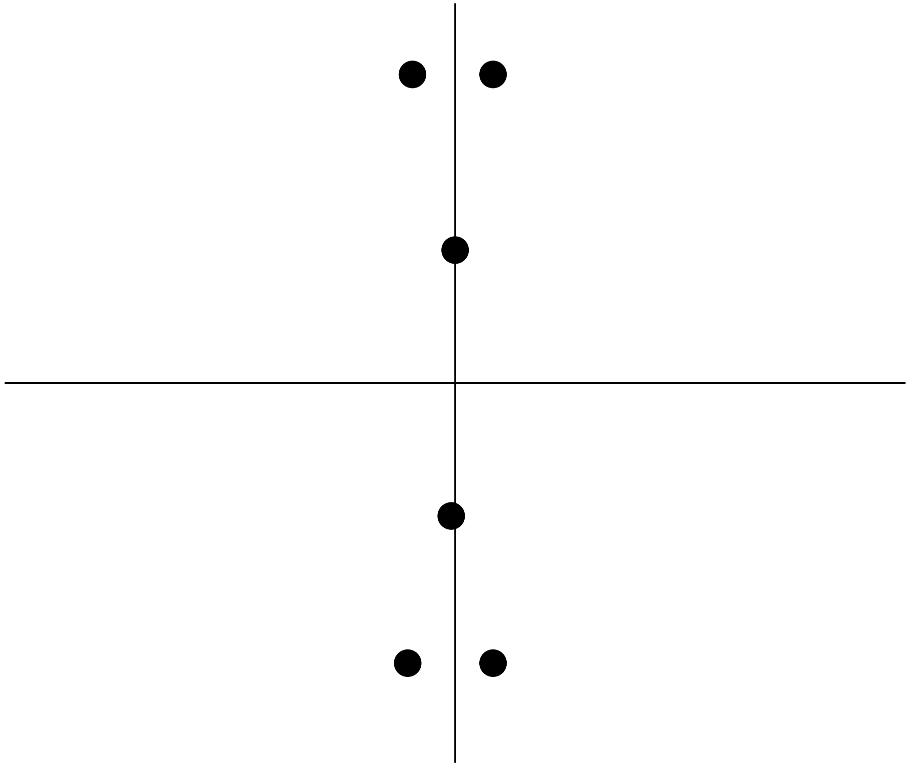


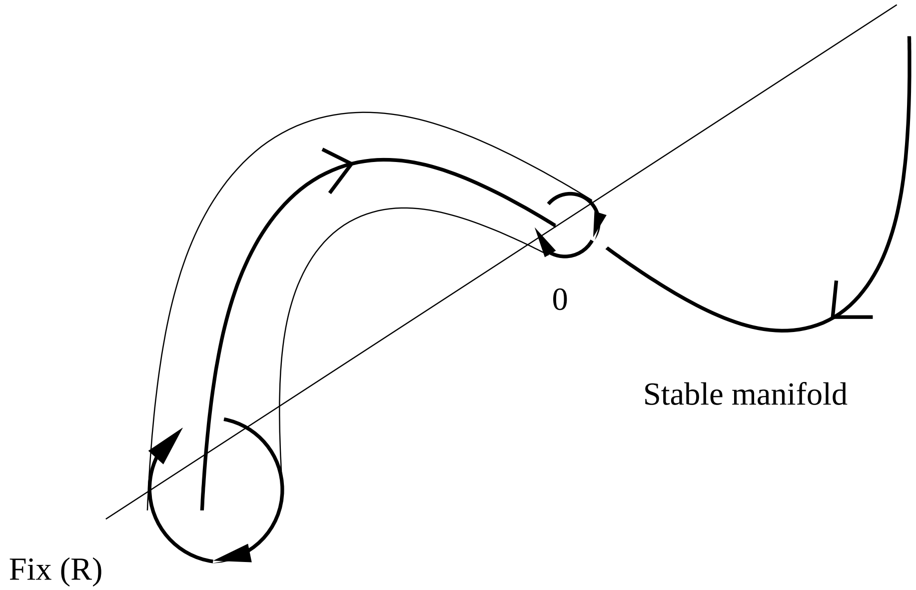
Case of the full normal form

Stable manifold $\cap$Fix(R) : $3N + 4$ conditions. $\Rightarrow$ codimension N $(N \geq p)$

Bifurcating homoclinic solutions

Reversible $(i\lambda_0)^2(i\lambda_1)$ resonance : $p = 1$ (travelling waves), $\tau \approx \tau_1 \Rightarrow N = 1$





E. Lombardi, Lecture Notes in Math. 1741 (2000) :

- The splitting distance between $W^s(0)$ and Fix(R) is exponentially small in $|\tau - \tau_1|$ (and does not generically vanish).
- There exist reversible solutions of the full normal form homoclinic to periodic orbits of sizes $O(e^{-c/|\tau-\tau_1|^{1/2}})$ $(c > 0)$.

Bifurcating homoclinic solutions

THEOREM 3 : Exact homoclinic solutions satisfying $u_{n+1}(t) = -u_n(t-\tau)$

Assume $p = 2$:

$$y_{n+2}(x) = y_n(x), \quad y_n(x) = u_n(\tau(n-x))$$

$$\frac{1}{\tau^2}\frac{d^2 y_n}{dx^2} = V'(y_{n+1}(x+1) - y_n(x)) - V'(y_n(x) - y_{n-1}(x-1)).$$

Assume $\tau \approx \tau_1$ with $\tau < \tau_1$, V even, $V^{(4)}(0) > 0$.

The problem is invariant under $-\sigma \ : \ (y_n) \mapsto -(y_{n+1})$.

On Fix$(-\sigma)$, the full normal form admits small amplitude reversible solutions homoclinic to periodic orbits.

For fixed τ (and up to a shift in x), these solutions occur in a one-parameter family parametrized by the amplitude of periodic orbits.

The lower bound of these amplitudes is $O(e^{-c/|\tau-\tau_1|^{1/2}})$ ($c > 0$).

= exact travelling breather solutions superposed at infinity on an oscillatory (periodic) tail.

6 – Some open problems

Some open problems

- Analytical study of global solution branches. Local results obtained for : $\tau \approx \tau_1$, $p = 1$, $p = 2$ and an even potential V.
- Codimension-1 case : existence of isolated values of τ such that the oscillatory tail vanishes ?
- Existence of homoclinic solutions to quasi-periodic orbits for N resonant modes, $N \geq 2$.
  Example : $p = 2$, non-even potential V, $\tau \approx \tau_1 \Rightarrow N = 2$.
- Incommensurate case $u_n(t) = u(n - t/\tau, t)$, with $u(x, t + T_b) = u(x, t)$, and $T_b/\tau \notin \mathbb{Q}$.
  In our case $T_b/\tau = p/m \in \mathbb{Q}$, and the dimension of the center manifold diverges as $p \to +\infty$.