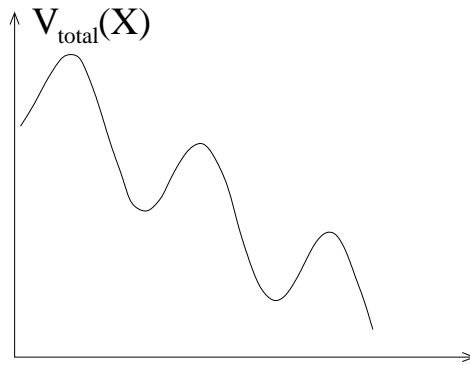


# **Fronts in "Rough Potentials"**

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# Systems with many local minima I



Magnetic domain walls under influence of random perturbations, phase change phenomena, dislocation lines in crystals in the presence of obstacles...

Here: Focus on a few "toy models"

Start with ODE-example:

$$V_\epsilon(x) := \frac{1}{2}x^2 + \epsilon \cos\left(\frac{x}{\epsilon}\right)$$

Convergence of energies:

$$\lim_{\epsilon \rightarrow 0} V_\epsilon(x) = V_0(x) = \frac{1}{2}x^2$$

## Systems with many local minima II

$$V_\epsilon(x) := \frac{1}{2}x^2 + \epsilon \cos\left(\frac{x}{\epsilon}\right)$$

Convergence of energies:

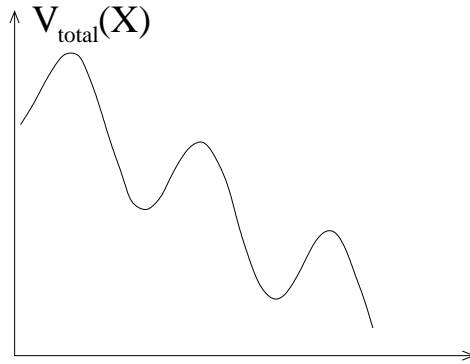
$$\lim_{\epsilon \rightarrow 0} V_\epsilon(x) = V_0(x) = \frac{1}{2}x^2$$

Gradients do not converge, there are stable equilibria ("Pinning States") for

$$\dot{x} = -V'_\epsilon(x)$$

Limit of energy and limit of gradient flow do NOT commute!

## Systems with many local minima III



"Escape" from local minima:

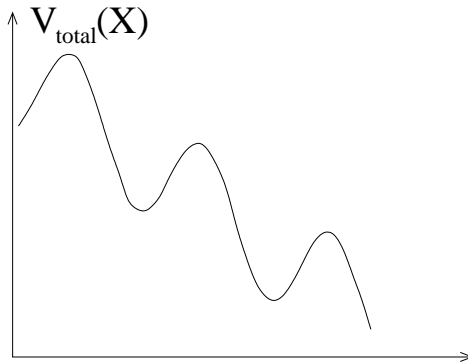
System subject to *external forces* and *thermal fluctuations*:

$$dX = -V'_\epsilon(x) + F + \sqrt{1/\beta}dW(t)$$

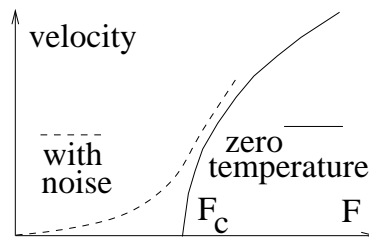
$F$  driving field,  $dW(t)$  thermal fluctuations,  $\beta$  inverse temperature

Measure **effective velocity** on macroscale.

## Systems with many local minima IV



$$dX = -V'_\epsilon(x) + F + \sqrt{\epsilon}dW(t)$$



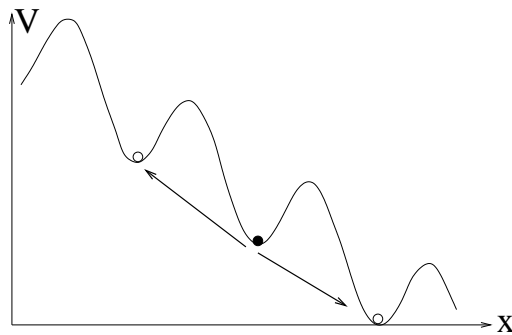
Effective velocity?

# Systems with many local minima **V**:

## Thermal case

(with N. Grunewald)

Blow-up:  $dX_C = C + \sin(X_C) + dW(t)$



Sample  $X_C(t)$  at wells  $\pi(2\mathbb{Z} + 1)$ . Let  $\tau_i$  be first time it hits the left or right neighbour of  $X_C(\tau_{i-1}) \Rightarrow$  Markov chain  $Y_i = X_C(\tau_i)$ .

Compute  $p_C := \mathbb{P}(Y_{i+1} = Y_i + 1) > 1/2$  and  $\mathbb{E}(\tau)$  from solution of "first exit problem,"

$$V_{eff}(C) = \lim_{T \rightarrow \infty} T^{-1} X_C(T) = (2p_C - 1) / \mathbb{E}(\tau).$$

## Systems with many local minima VI

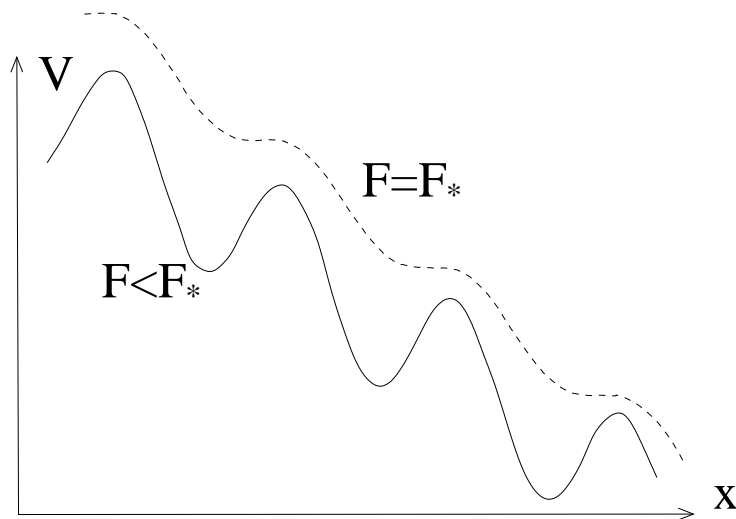
Now: Zero temperature

Rescale again and consider

$$\dot{X} = \sin(X) + F.$$

$F \leq 1 = F_*$  : Stationary solutions

$F > F_*$  : Time-periodic solution



## Systems with many local minima VII

Effective velocity

$$V_{eff} = \lim_{T \rightarrow \infty} \frac{X(T)}{T}$$

Linearize

$$\dot{X} = \sin(X) + F$$

around  $X_0 = 3\pi/2$ , let  $F = 1 + \gamma$ :

$$\dot{Y} = \gamma + \frac{1}{2}Y^2, \quad Y = X - X_0.$$

Solution:

$$Y(t) = \sqrt{2\gamma} \tan\left(\sqrt{(\gamma/2)} t\right)$$

## PDE-Model: The Energy

Study the following functional: (Joint work with M. Lucia und M. Novaga)

$$G_\epsilon(u) := \int_{\Omega} \left[ \epsilon |\nabla u|^2 + \frac{W(u)}{\epsilon} + \frac{1}{\epsilon^\alpha} f\left(\frac{x}{\epsilon^\alpha}\right) u \right] dx,$$

$f$  periodic,  $\langle f \rangle = 0$ ,  $\alpha > 0$ ,  $f$  symmetric under reflection at coordinate axis:  $\Rightarrow$  Minimizer of  $G$  on a cell is periodic, but not constant.

$\inf_{H^1(\Omega)} G_\epsilon(m, \Omega) \rightarrow -\infty \Rightarrow$  Renormalization:

$$F_\epsilon(u) = \underbrace{G_\epsilon(u)}_{\text{two phases}} + \underbrace{|c_\epsilon| |\Omega|}_{\text{pure phase}}$$

Alternative Formulation ( $\Gamma$ -expansion):

$$\epsilon G_\epsilon \sim \bar{W} + \epsilon^{2-2\alpha} c^{(1)}(1 + o(1)) + \epsilon F_0$$

$$G_\epsilon(u) := \int_{\Omega} \left[ \epsilon |\nabla u|^2 + \frac{W(u)}{\epsilon} + \frac{1}{\epsilon^\alpha} f\left(\frac{x}{\epsilon^\alpha}\right) u \right] dx,$$

$$F_\epsilon(u) = \underbrace{G_\epsilon(u)}_{\text{two phases}} + \underbrace{|c_\epsilon| |\Omega|}_{\text{pure phase}}$$

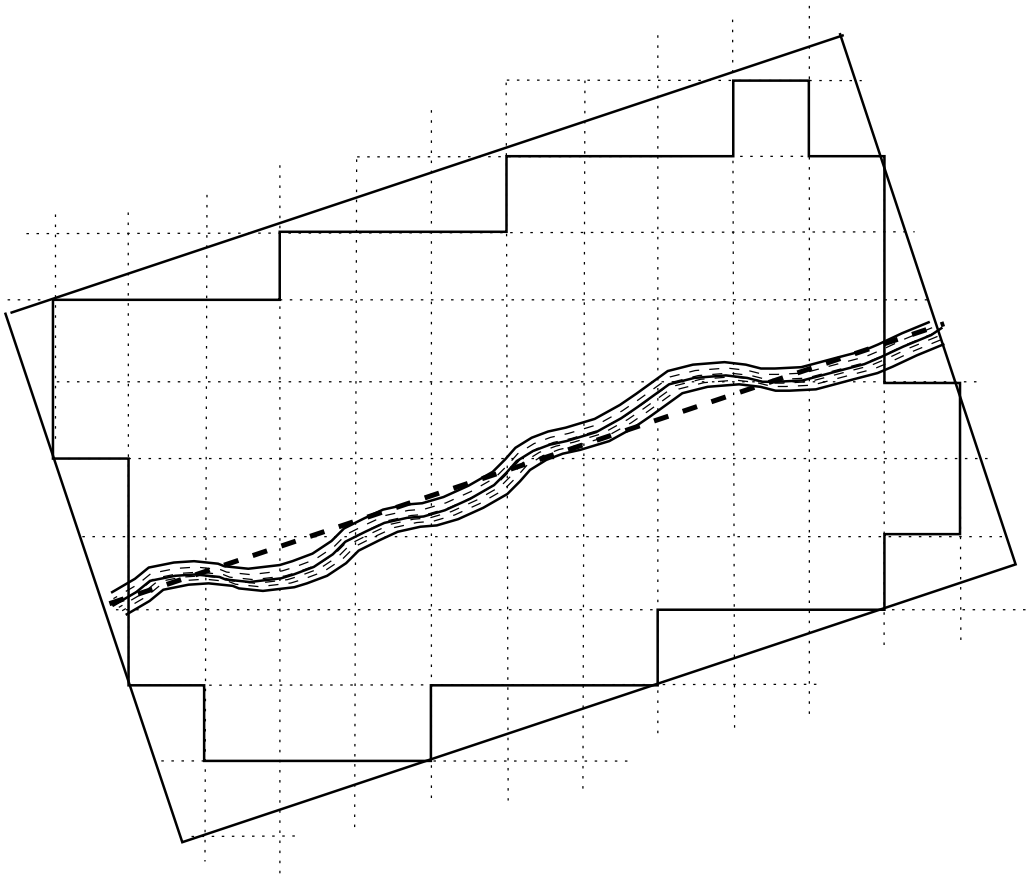
$\alpha < 1$ , or  $\alpha = 1$  and additional conditions:

$$\Gamma - \lim_{\epsilon \rightarrow 0} F_\epsilon(u) = F_0(u) \quad (L^1 - \text{metric})$$

$$F_0(u) = \begin{cases} \int_{\Omega} \varphi(\nabla u / |\nabla u|) |\nabla u| (dx) \\ \text{for } u \in BV(\Omega), |u| = 1 \text{ a.s.} \\ +\infty \quad \text{else} \end{cases}$$

anisotropic surface energy, cf. Ansini-Braides-Chiado Piat

$\alpha < 1$  : separation of scales: compute  $\varphi$  with the help of a sharp interface functional.



## Key Lemma

Now:  $0 < \alpha < 1$ . Blow-up:  $\epsilon^{-\alpha}x \rightarrow x$ ,  $\epsilon^{1-\alpha} \rightarrow \epsilon$

$$\tilde{G}(u, \Omega) := \int_{\Omega} \left( \epsilon |\nabla u|^2 + \frac{W(u)}{\epsilon} \right) dx + \int_{\Omega} f u dx$$

If  $\|f\|_{L^n}$  is small enough, and  $W$  "symmetric" around wells, then  $\tilde{G}$  has exactly *two* absolute minimizers  $u_{\epsilon}^{\pm}$  such that:

$u_{\epsilon}^{\pm}$  *periodic*,

$$\|u_{\epsilon}^{\pm} - (\pm 1)\|_{\infty} = \mathcal{O}(\epsilon)$$

$$\begin{aligned} & \tilde{G}(u, \Omega) - \min \left( \tilde{G}(u_{\epsilon}^{+}, \Omega), \tilde{G}(u_{\epsilon}^{-}, \Omega) \right) \\ & \geq C \int_{-1/2}^{1/2} P(\{u < s\}, \Omega) ds. \end{aligned}$$

## Consequence

Assume

$$\limsup_{\epsilon \rightarrow 0} F_\epsilon(u_\epsilon) < \infty.$$

Then

(a) If  $u_\epsilon \rightarrow u$  in  $L^1(\Omega)$ , then  $|u| = 1$  a.s.;

(b) There exists  $u \in BV(\Omega)$  with  $|u| = 1$  a.e.,  
such that (for subsequence)

$$\|u_\epsilon - u\|_{L^1} \rightarrow 0,$$

$$\int_{\Omega} |\nabla u| \leq C \liminf_{\epsilon \rightarrow 0} F_\epsilon(u_\epsilon, \Omega).$$

## Time evolution/relaxation:

Surface gets stuck in local minima!

Evolution formally

$$V = c_W \kappa - \frac{1}{\epsilon} f \left( \frac{x}{\epsilon} \right)$$

"Strong" forcing,  $f$  changes sign,  $\Rightarrow$  Pinning!

Add external field: De-Pinning

Homogenization results: Bhattacharya/Craciun and Lions/Souganidis (No change of sign, no pinning)

Lattice models: E.g. Cahn, Mallet-Paret and Van Fleck, Carpio and Bonilla

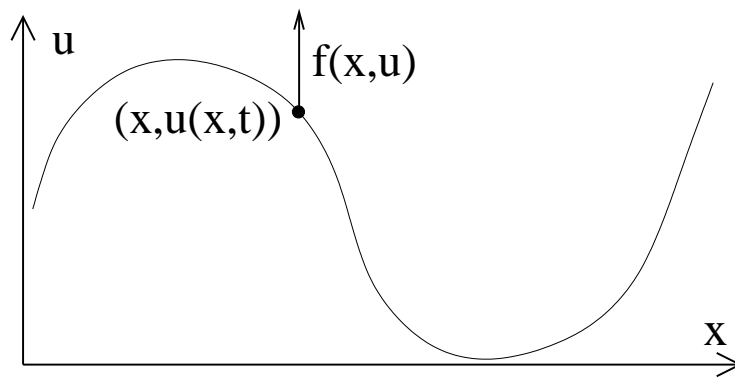
## Pinning/De-Pinning

Here we make simplifying assumptions:

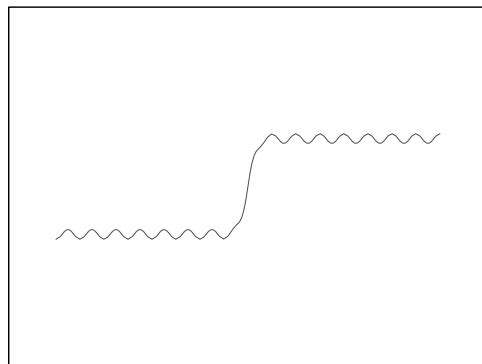
**Case 1** Surface is graph, linearize around plane:

$$u_t = u_{xx} - f(u, x)$$

Surface: Graph of  $u(x, t)$



**Case 2** Consider Allen-Cahn equation in 1 d



## “Pinning” and “De-Pinning” of Fronts (With A.N.K. Yip)

$u(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$  periodic in  $x$  solves

$$u_t = \Delta u + f(x, u) + F$$

$f(\cdot, \cdot)$  is 1-periodic in *both* variables, sufficiently regular (bounded), not constant in  $u$  and

$$\int_0^1 \int_0^1 f(x, u) du dx = 0$$

$F \geq 0$  : external driving force.

Example:  $f(x, u) = \sin(x) \cos(u)$

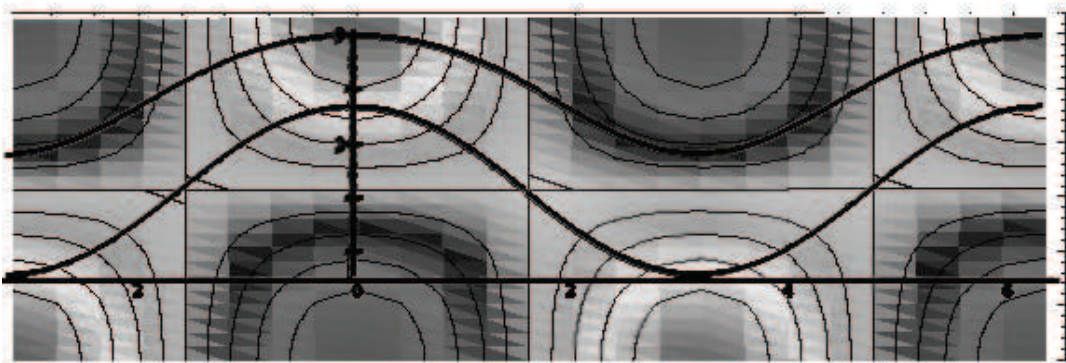
## Pinning for small external driving force

For  $F = 0$  there exists a stationary solution. Assume that it is *stable*.

Then there exists  $F_* > 0$  such that for any  $0 \leq F \leq F_*$  there exists a periodic stationary solution  $u_F$ , i.e. a space periodic solution of

$$\Delta u_F + f(\cdot, u_F) + F = 0.$$

Comparison principle:  $L^\infty$ -initial data pinned.



## Pulsating Wave for large driving force

For  $F > F_*$ , there exist *pulsating wave solutions*  $U_F(x, t)$  with velocity  $V_F$  :

$U_F(x, t)$  is periodic in  $x$ , solves

$$(u_F)_t = \Delta u_F + f(x, u) + F$$

and has the property

$$U_F(\cdot, t + 1/V_F) = U_F(\cdot, t) + 1.$$

## **Pulsating Wave for large driving force**

Comparison principle:

Pulsating wave  $U_F$  controls the asymptotic speed for trajectories starting from  $L^\infty$  initial data.

So its speed  $V_F$  is the asymptotic speed of the front.

Estimate  $V_F$ ?

## Scaling of the speed for $F$ near $F_*$

For  $F = F_*$  stationary solution  $u_*$  exists. Let

$$L(u_*)\varphi = \Delta\varphi + (\partial_u f)(\cdot, u_*)\varphi, \quad \varphi(x) \text{ periodic.}$$

Then  $L(u_*)$  has principal (simple) eigenvalue 0 with positive eigenfunction  $\varphi_p$ .

Under the *non-degeneracy* condition

$$\int_{[0,1]^n} \varphi_p^3(x) (\partial_u^2 f)(x, u_*(x)) dx \neq 0 \quad (*)$$

we obtain that

$$V_F = A(F - F_*)^{\frac{1}{2}} + o(|F - F_*|^{\frac{1}{2}})$$

for a constant  $A$  depending on  $(*)$ .

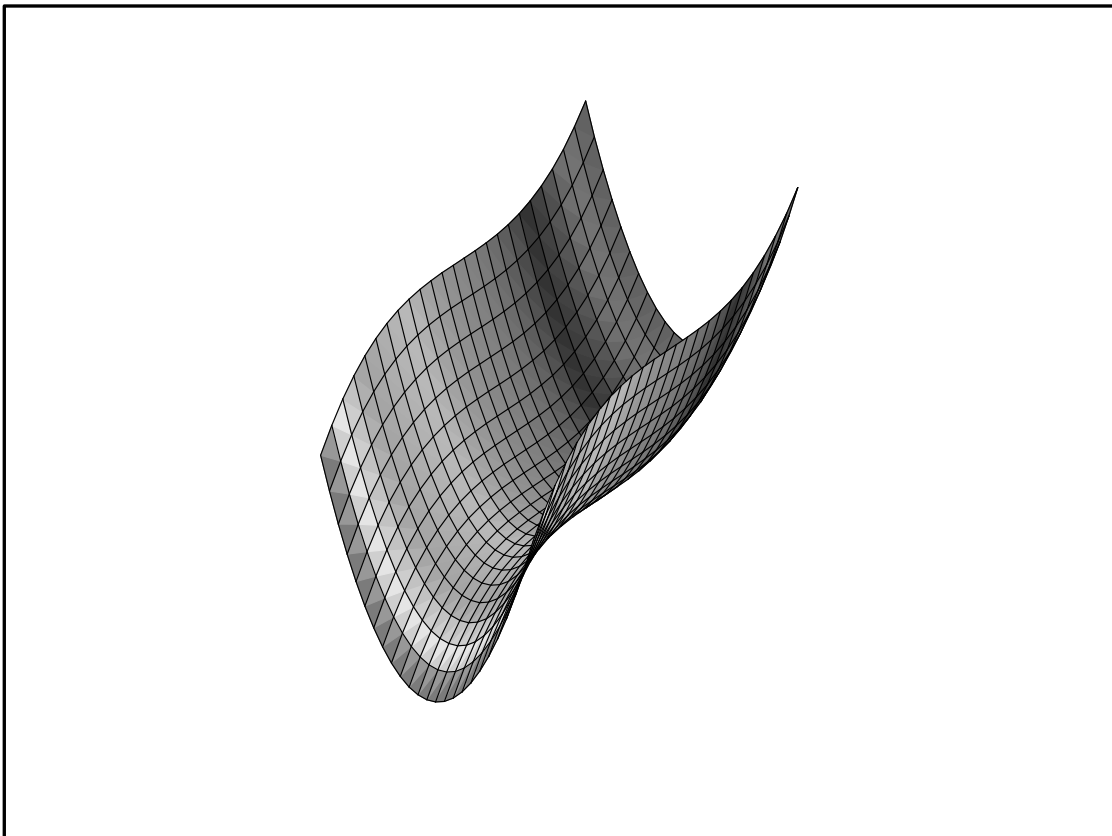
## Non-Degeneracy condition

$$A := \int_{[0,1]^n} \varphi_p^3(x) (\partial_u^2 f)(x, u_*(x)) dx \neq 0$$

Energy in direction of principal eigenfunction:

$$E(u_* + s\varphi_p) = -s^3 A + o(s^3)$$

Spectral gap: Contraction in other directions



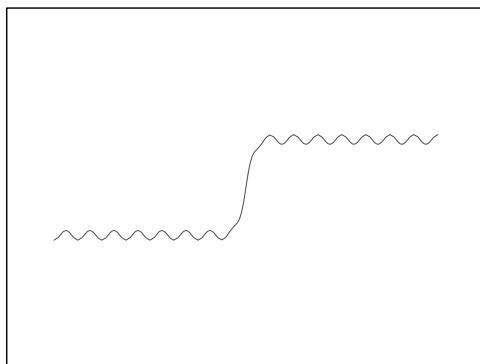
## Setting for 1-d Allen-Cahn equation (perturbative regime)

$$v_t = v_{xx} - \frac{1}{2}W'(x) + \delta(g(x) + F), \quad x \in \mathbb{R}$$

where  $0 < \delta \ll 1$ ,  $g$  smooth, 1-periodic, mean zero, and  $W(x)$  is a double-well potential.

End states: For  $\delta$  small, there are exactly two 1-periodic stationary solutions close to  $\pm 1$ .

*Front*: Solution which connects the two end states. (Convergence exponentially fast)



As before  $0 < F \leq F_*$  : Stationary front exists.

## De-Pinning for 1-d Allen-Cahn equation (perturbative regime)

$$v_t = v_{xx} - \frac{1}{2}W'(x) + \delta(g(x) + F), \quad x \in \mathbb{R}$$

Perturbation around  $\delta = 0$  :

For  $\delta = 0$  Instanton (standing wave)  $m$  solves

$$m_{xx} = \frac{1}{2}W'(m), \quad m(0) = 0, \quad \lim_{x \rightarrow \pm\infty} m = \pm 1$$

## De-Pinning for 1-d Allen-Cahn equation (perturbative regime)

Consider

$$h(a) = - \int_{\mathbb{R}} g(x+a)m_x(x)dx$$

("Force"), and assume  $h''(0) > 0$ . ( $m$  is the standing wave centered at 0)

Then for  $F - F_* > C\delta$  and  $\delta$  sufficiently small:

There exists pulsating wave  $U_F(x, t)$  such that

$$U_F(x, t + 1/V_F) = U_F(x - 1, t)$$

and

$$V_F \sim \delta \sqrt{F - F_*}.$$

## Asymptotics of speed for 1-d Allen-Cahn equation

$$(U_F)_t = (U_F)_{xx} - \frac{1}{2}W'(U_F) + \delta(g(x) + F),$$
$$U_F(x, t + 1/V_F) = U_F(x - 1, t)$$

Precise asymptotics for  $\delta \rightarrow 0$  :

$$V_F = \delta \frac{\alpha}{\pi} \sqrt{\beta(F - F_*)} + o\left(\delta \sqrt{(F - F_*)}\right),$$

$$\alpha = \frac{2}{\|m_x\|_2^2} \text{ mobility}, \quad \beta = \frac{h''(0)}{4\|m_x\|_2^2}$$

## Sketch of the proof I

Ingredients:

- Comparison principle: Time evolution respects order.
- Parabolic regularity
- Spectral properties of linearisation at
  - $u_*$  in the graph case
  - instanton  $m_x$  in the Allen-Cahn case
- construction of sub/supersolutions

## Sketch of the proof II

- Schauder's fixed point theorem: Existence of a solution of

$$u_t = \Delta u + f(x, u) + F$$

is known, we have to show existence of *pulsating wave*. Show:

$$u \rightarrow u(x, T(u)) - 1$$

has fixed point, where

$$T(u) := \inf\{t > 0 : \langle u(\cdot, t) \rangle > \langle u(\cdot, 0) \rangle + 1\}$$
$$\langle v \rangle = \int_{[0,1]^n} v(x) dx.$$