Models of Ferromagnetic Hysteresis

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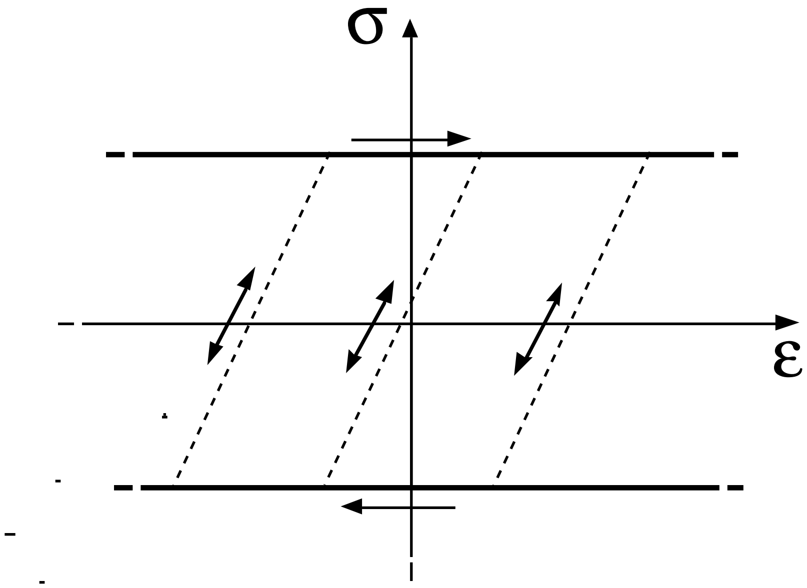
Plan

Continuous and discontinuous hysteresis

Relay and Preisach model

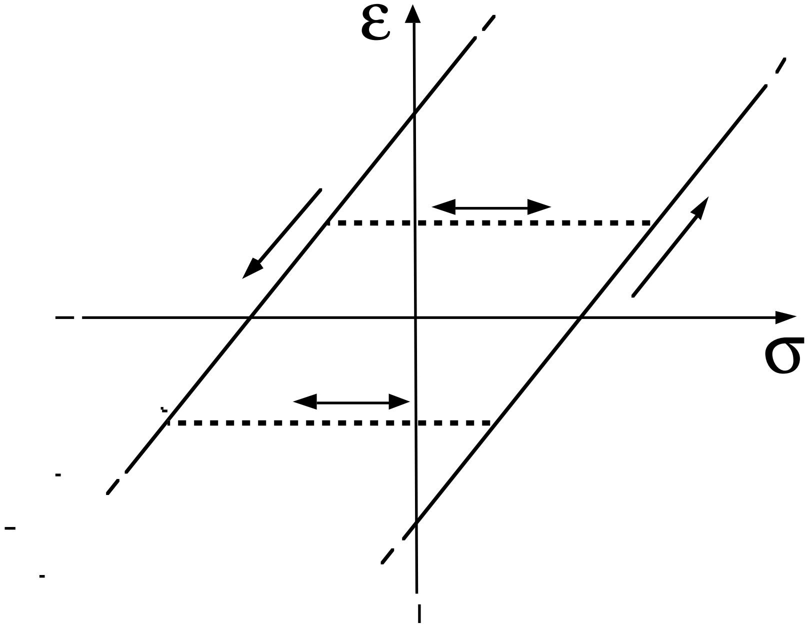
Scalar and vector Maxwell system with hysteresis

Continuous Hysteresis: examples from elasto-plasticity



Stop

$$\sigma' + \mathrm{sign}^{-1}(\sigma) \ni \varepsilon'$$



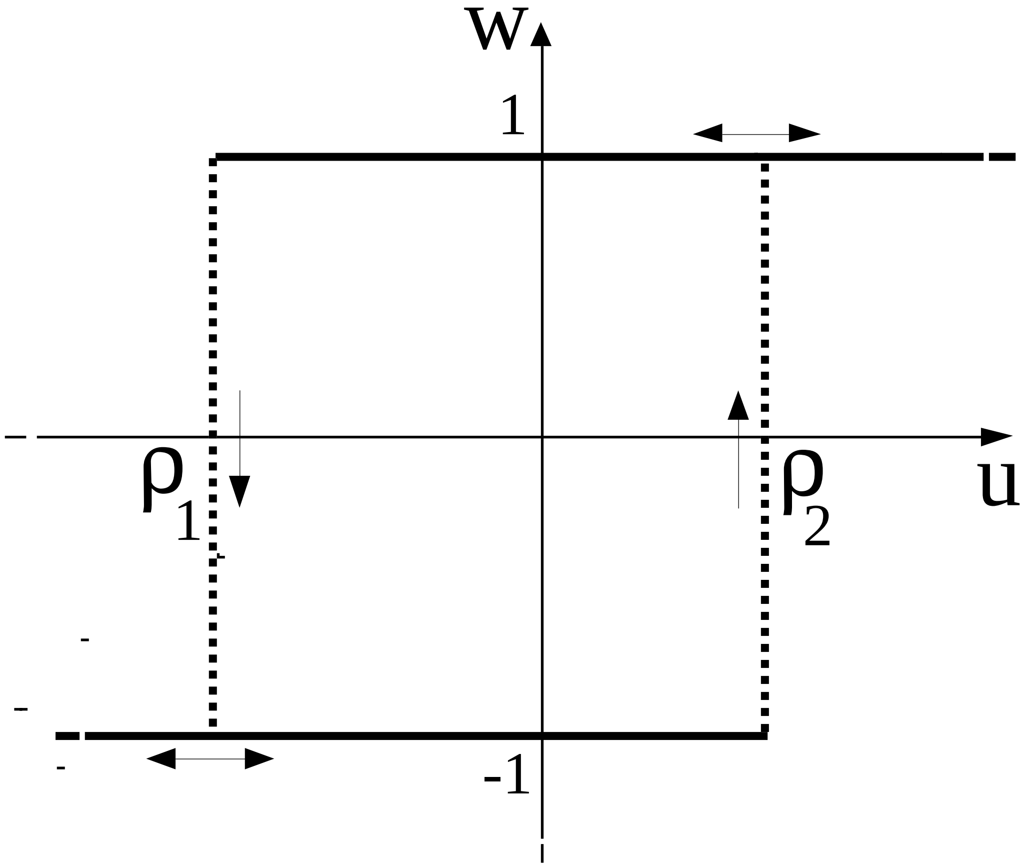
Play

$$\mathrm{sign}(\varepsilon') + \varepsilon \ni \sigma$$

The large class of **Prandtl-Ishlinskiĭ models** is obtained by combining stops and plays in parallel and in series (respectively): ... is this construction really *legitimate*?

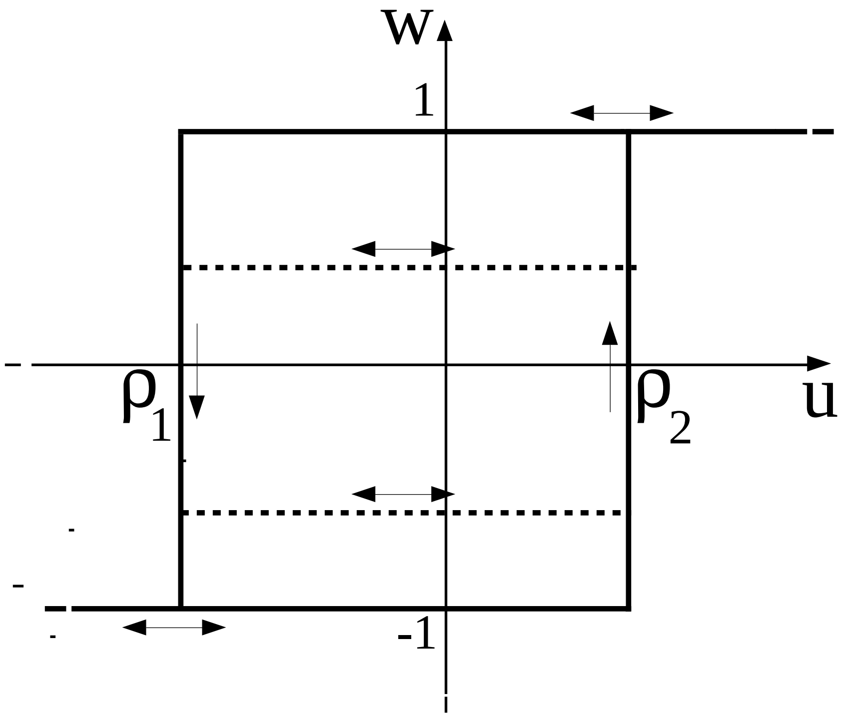
Discontinuous Hysteresis

Relay. For any pair $\rho := (\rho_1, \rho_2) \in \mathbf{R}^2$ $(\rho_1 < \rho_2)$, we define the *relay operator* h_ρ:



The operator $h_\rho : C^0([0,T]) \times \{-1,1\} \to BV(0,T)$ is not closed.

In connection with P.D.E.s, it is of interest to deal with the closure of h_ρ.



(i) *Confinement condition:*

$$\begin{cases} |w| \le 1 \\ (w-1)(u-\rho_2) \ge 0 \\ (w+1)(u-\rho_1) \ge 0 \end{cases}$$

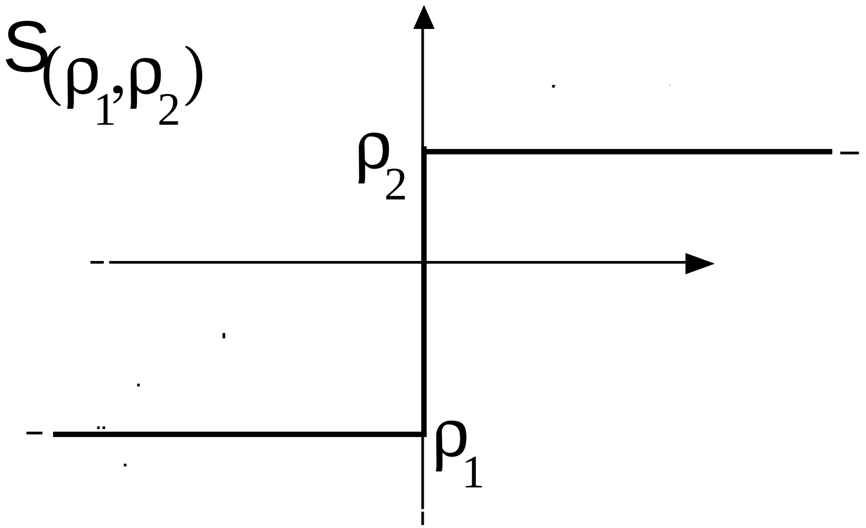
(ii) *Dissipation condition:*

$$\begin{cases} \displaystyle\int_0^t u\,dw \ge \int_0^t \Big[\rho_2 (dw)^+ - \rho_1 (dw)^-\Big] \\ \displaystyle\qquad = \frac{\rho_2-\rho_1}{2}\int_0^t |dw| + \frac{\rho_2+\rho_1}{2} w\Big|_0^t \\ \qquad =: \varPsi_\rho(w,t) \qquad \forall t \end{cases}$$

The relay $u \mapsto w$ may also be represented by a *double variational inequality:*

$$S_{(\rho_1,\rho_2)}(w') + S^{-1}_{(-1,1)}(w) \ni u.$$

This inclusion is easily studied for non-space-distributed systems.



At variance with stops and plays,

relays cannot be represented by a **single** variational inequalities.

A new model of **quasi-static** evolution with hysteresis:

stability condition + energy balance

may be compared with the *confinement* and *dissipation* conditions above.

— A. Mielke, F. Theil, V. Levitas: *A variational formulation of rate-independent phase transformations using an extremum principle.*
Arch. Rat. Mech. Anal. 162 (2002) 137–177
— A. Mielke: *Analysis of energetic models for rate-independent materials.*
In Proceedings of the I.C.M., Vol. III (Beijing, 2002)
— A. Mielke, F. Theil: *On rate-independent hysteresis models.*
Nonl. Diff. Eqns. Appl. 11 (2004) 151–189.

Open question: extension to **non-quasi-static** evolution.

A similar model had been proposed for **quasi-static** brittle fracture dynamic:

— G.A. Francfort, J.-J. Marigo:
Revisiting brittle fracture as an energy minimization problem...
J. Mech. Phys. Solids 46 (1998) 1319-1342

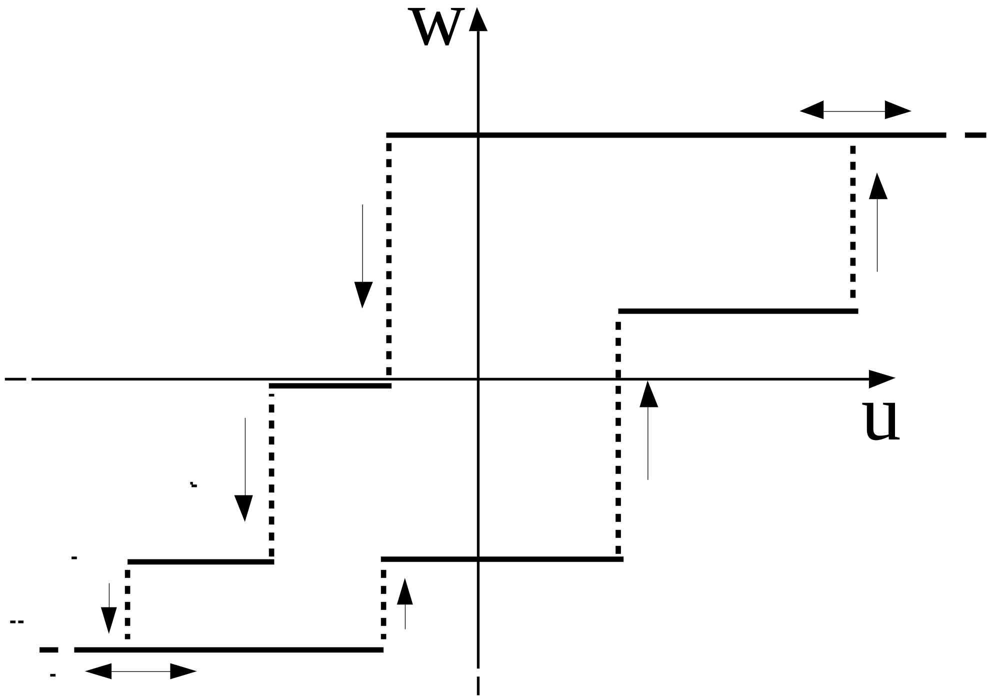
See also

— G. Dal Maso, R. Toader:
A model for the quasi-static growth or brittle fractures...
Arch. Rat. Mech. Anal. 162 (2002) 101-135
— G.A. Francfort, L.C. Larsen:
Existence and convergence for quasi-static evolution in brittle fracture.
Comm. Pure Appl. Math. 56 (2003) 1465–1500
— G.A. Francfort, G. Dal Maso, R. Toader:
Quasi-static crack growth in nonlinear elasticity.
Arch. Rat. Mech. Anal. 176 (2005) 165-226

Preisach's Model (1935)

Linear combination of delayed relays with different thresholds and *the same input:*

$$\mathcal{H}_\mu(u, \{\xi_\rho\}) := \int_{\rho_1 < \rho_2} h_\rho(u, \xi_\rho)\, d\mu(\rho) \qquad \text{in } [0, T].$$



Under natural hypotheses on the *Preisach measure* μ,

$\mathcal{H}_\mu$ operates and is continuous in $C^0([0, T])$.

Scalar Quasilinear Hyperbolic Equation with Hysteresis

$$\frac{\partial^2}{\partial t^2}[u + \mathcal{F}(u)] - \Delta u = f \qquad (\mathcal{F} = \text{ relay}).$$

1. Continuous hysteresis operator $\mathcal{F}$

- Existence of a periodic solution for $\mathcal{F}$ = scalar *Prandtl-Ishlinskiĭ model* in 1 dim.:
  - — P. Krejčí: *Hysteresis and periodic solutions of semi-linear and quasi-linear wave equations.* Math. Z. 193 (1986) 247–264
- Well-posedness for $\mathcal{F}$ = tensor *Prandtl-Ishlinskiĭ model* in N dim.:
  - — A. V.: *Rheological models and hysteresis effects.* Rend. Sem. Mat. Univ. Padova 77 (1987) 213–243
- Well-posedness, existence of periodic solution, and asymptotic behaviour, in the convexity region of a continuous scalar hysteresis operator $\mathcal{F}$ in 1 dim.:
  - — P. Krejčí: *A monotonicity method for solving hyperbolic problems with hysteresis.* Apl. Mat. 33 (1988) 197–203
  - — other papers of Krejčí (cf. monograph).

2. Discontinuous hysteresis operator: $\mathcal{F}$ = relay or Preisach.

Problem 1. *To find* U *and* w *such that, setting* $u := \frac{\partial U}{\partial t}$,

$$\frac{\partial}{\partial t}(u + w) - \Delta U = F \qquad \textit{in } \mathcal{D}'(Q)$$

$$|w| \leq 1, \qquad \begin{cases} (w-1)(u-\rho_2) \geq 0 \\ (w+1)(u-\rho_1) \geq 0 \end{cases} \qquad \textit{a.e. in } Q$$

$$\frac{1}{2}\int_\Omega \left[u(x,t)^2 - u^0(x)^2 + |\nabla U(x,t)|^2\right] dx + \int_\Omega \Psi_\rho(w(x,\cdot),t) \leq \int_0^t \langle F, u\rangle \, d\tau \quad \textit{for a.a. } t$$

$$\gamma_0 U = 0 \quad \textit{on } (\Omega \times \{0\}) \cup (\partial\Omega \times]0,T[)$$

$$(u + w)|_{t=0} = u^0 + w^0 \qquad \textit{in } \Omega.$$

Theorem 1. $F \in L^1_t(L^2_x) + W^{1,1}_t(H^{-1}_x) \;\Rightarrow\; \exists$ *solution* (U, w)*:*

$$U \in W^{1,\infty}\big(0,T;L^2(\Omega)\big) \cap L^\infty\big(0,T;H^1_0(\Omega)\big).$$

This can be extended to the Preisach model. The argument is based upon:

(i) approximation via implicit time-discretization,

(ii) derivation of a priori estimates; in particular, by the dissipation condition,

$$\Big\|\frac{\partial w_m}{\partial t}\Big\|_{C^0(\bar{Q})'} = \int_\Omega dx \int_0^T |dw_m| \leq \text{Constant},$$

(iii) passage to the limit by compactness and lower semicontinuity.

A. V.: *Quasi-linear hyperbolic equations with hysteresis.*
Ann. Inst. H. Poincaré. Nonlinear Analysis, **19** (2002), 451-476

The argument also uses the following *compensated compactness* result.

Lemma. *If*

$$\begin{aligned} &u_m \to u && \textit{weakly in } L^2(Q) \cap H^{-1}\big(0,T;H^1(\Omega)\big) \\ &w_m \to w && \textit{weakly star in } L^\infty(Q) \\ &\left\| \frac{\partial w_m}{\partial t} \right\|_{L^1(Q)} \le \text{Constant} \end{aligned}$$

then

$$\iint_Q w_m u_m \, dxdt \to \iint_Q wu \, dxdt.$$

Well-posedness, existence of periodic solution, and asymptotic behaviour, in the convexity region of a continuous scalar hysteresis operator $\mathcal{F}$ in 1 dim.:

P. Krejčí: *A monotonicity method for solving hyperbolic problems with hysteresis.* Apl. Mat. 33 (1988) 197–203

Vector Problem — Maxwell-Ohm’s Equations

$$c\nabla\times\vec{H} = 4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad c\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad (\nabla\times := \text{curl})$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\cdot\vec{D} = 4\pi\hat{\rho} \qquad (\nabla\cdot := \text{div})$$

Ohm’s law: $\vec{J} = \sigma\vec{E} + \vec{J}_e$ Dielectric relation: $\vec{D} = \epsilon\vec{E}$.

$$\Rightarrow \qquad \epsilon\frac{\partial^2\vec{B}}{\partial t^2} + 4\pi\sigma\frac{\partial\vec{B}}{\partial t} + c^2\nabla\times\nabla\times\vec{H} = 4\pi c\nabla\times\vec{J}_e \ \ (\text{: datum}).$$

In ferr<u>i</u>magnetic insulators: $\sigma = 0 \quad \to$ quasilinear hyperbolic

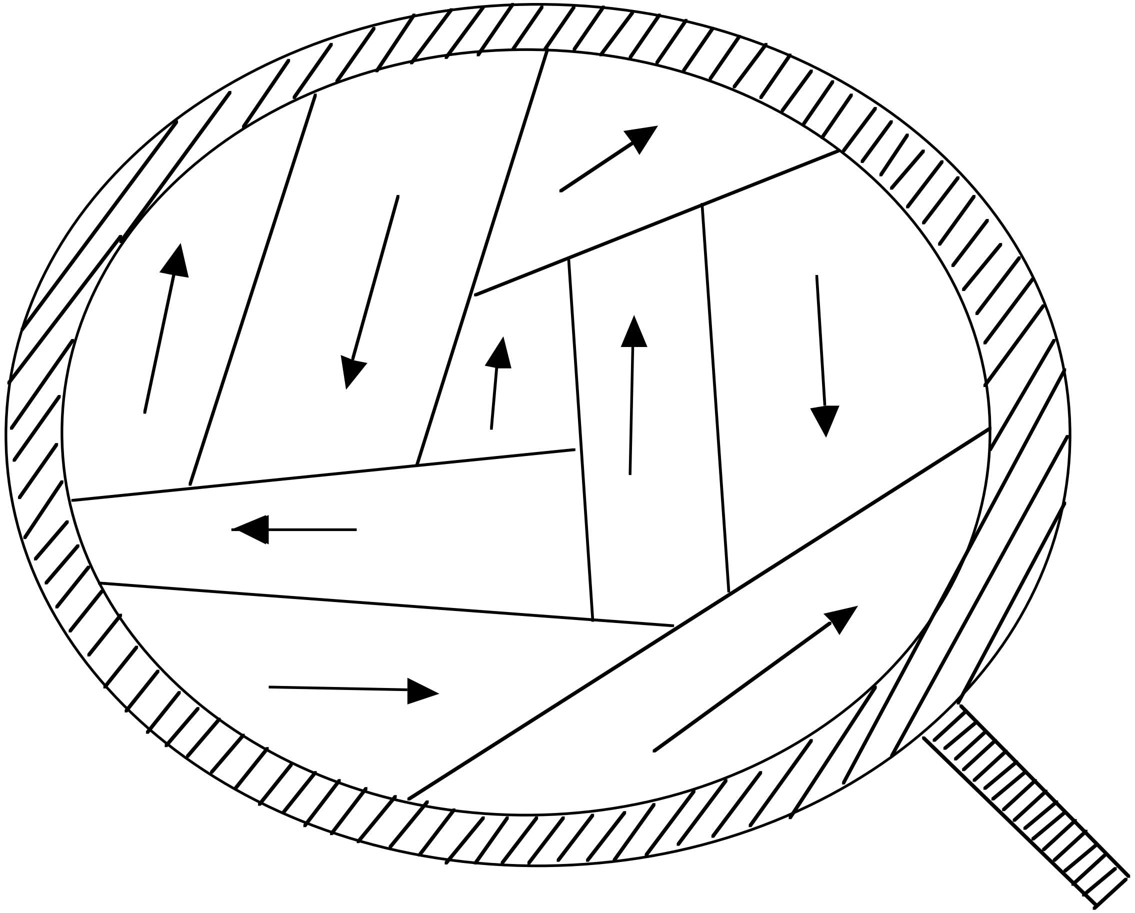
In ferr<u>o</u>magnetic metals: $\epsilon\left|\frac{\partial^2\vec{B}}{\partial t^2}\right| \ll 4\pi\sigma\left|\frac{\partial\vec{B}}{\partial t}\right| \ \to$ quasilinear parabolic

For quasi-static processes: $c\nabla\times\vec{H} = 4\pi\vec{J} \qquad \nabla\cdot\vec{B} = 0.$

Constitutive Law $\vec{H} \mapsto \vec{M}$

At any macroscopic point sits a population of *magnetic domains*, characterized by

(i) a magnetization direction $\vec{\theta} \in S^2$, (ii) a pair of thresholds $\rho := (\rho_1, \rho_2) \in \mathcal{P}$.



Vector Relay. The *vector relay* $\vec{h}_{(\rho,\vec{\theta})}$ is defined in terms of the scalar relay h_ρ:

$$\vec{h}_{(\rho,\vec{\theta})}(\vec{H}) := h_\rho(\vec{H}\cdot\vec{\theta})\vec{\theta} \qquad \forall(\rho,\vec{\theta}) \in \mathcal{P} \times S^2.$$

This may represent the behaviour of a crystal having crystallographic orientation $\vec{\theta}$.

Theorem 2. *Each of the above 3 P.D.E. systems has a weak solution if coupled with*

$$\vec{M} = h_\rho(\vec{H}\cdot\vec{\theta})\vec{\theta} \qquad \textit{pointwise in } Q.$$

A. V.: Maxwell's equations with vector hysteresis.

Archive Rat. Mech. Anal. **175** (2005) 1–38

Vector Preisach Model

μ: finite positive Borel measure on $\mathcal{P} \times S^2$

$$\vec{M} = \vec{\mathcal{F}}_\mu(\vec{H}) := \iint_{\mathcal{P}\times S^2} \vec{h}_{(\rho,\vec{\theta})}(\vec{H})\, d\mu(\rho,\vec{\theta}).$$

A. Damlamian, A. V.:
Une généralisation vectorielle du modèle de Preisach pour l'hystérésis.
C.R. Ac. Sci. Paris, I **297** (1983) 437–440

I.D. Mayergoyz: *Vector Preisach model of hysteresis.*
J. Appl. Phys. **63** (1988) 2995–3000

The extension of the existence results to the Maxwell system coupled with the vector Preisach model is an **open question.**

The Preisach model is macroscopic and phenomenologic.

See the Landau-Lifshitz-Brown theory of **micromagnetism**

for an analysis of the *microscopic structure.*

L. Landau, E. Lifshitz:
On the theory of dispersion of magnetic permeability in ferromagnetic bodies.
Physik. Z. Sowietunion **8** (1935) 153–169

W.F. Brown jr.:
Micromagnetics. Interscience, New York 1963

Some Monographs on Hysteresis

[1] G. Bertotti: *Hysteresis in magnetism.* Academic Press, Boston 1998

[2] M. Brokate, J. Sprekels: *Hysteresis and phase transitions.* Springer, Berlin 1996

[3] M.A. Krasnosel’skiĭ, A.V. Pokrovskiĭ: *Systems with hysteresis.* Springer, Berlin 1989 (Russian ed. Nauka, Moscow 1983)

[4] P. Krejčí: *Convexity, hysteresis and dissipation in hyperbolic equations.* Gakkotosho, Tokyo 1997

[5] I.D. Mayergoyz: *Mathematical models of hysteresis and their applications.* Elsevier, Amsterdam 2003

[6] A. Visintin: *Differential models of hysteresis.* Springer, Berlin 1994

In press:

[7] G. Bertotti, I.D. Mayergoyz (Eds.): *The science of hysteresis.* Elsevier 2005