THE HEAT EQUATION IN $L_q((0,T),L_p)$-SPACES WITH WEIGHTS

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1. Introduction

We are going to investigate the equation

$$u_t(t,x)=a^{ij}(t)u_{x^ix^j}(t,x)+f(t,x) \tag{1.1}$$

in the domains

$$\mathsf{R}\times\mathsf{R}^d=\{(t,x):t\in\mathsf{R},x\in\mathsf{R}^d\},\quad \mathsf{R}\times\mathsf{R}^d_+=(\mathsf{R}\times\mathsf{R}^d)\cap(x^1>0),\quad \mathsf{R}_+\times\mathsf{R}^d,\quad \mathsf{R}_+\times\mathsf{R}^d_+,$$

where a^{ij} are bounded measurable functions of $t\in\mathsf{R}$ satisfying the uniform ellipticity condition:

$$K|\lambda|^2\ge a^{ij}(t)\lambda^i\lambda^j\ge\delta|\lambda|^2. \tag{1.2}$$

Equation (1.1) is understood in the sense of generalized functions only with respect to x. In other words, say in the case of the whole $\mathsf{R}\times\mathsf{R}^d$ by a solution of (1.1) we mean a function $u(t)$, $t \in \mathsf{R}$, taking values in the set of generalized functions on R^d such that, for any $t, s \in \mathsf{R}$ satisfying $t \geq s$ and test function $\varphi \in C_0^\infty(\mathsf{R}^d)$, we have

$$(u(t), \varphi) = (u(s), \varphi) + \int_s^t [a^{ij}(r)(u(r), \varphi_{x^i x^j}) + (f(r), \varphi)]\, dr.$$

The emphasis is on proving solvability in function spaces of Sobolev type with different powers of summability p and q with respect to x and t. This issue arose from the theory of stochastic partial differential equations in domains in Sobolev spaces with weights and it turns out that, in this theory, the spaces with weights are the only reasonable ones where to look for solutions to equations in domains. To illustrate this necessity consider

$$u_t = u_{xx} \quad \text{in} \quad R^2_+ \quad \text{with} \quad u(0, x) = 0, u(t, 0) = g(t),$$

where g is a bounded *nondifferentiable function*. Then u_{xx} cannot be continuous since otherwise

$$0 = u(t, 0) - u(0, 0) = g(t) - g(0) = \int_0^t u_{xx}(s, 0)\, ds.$$

This equation arrises from the SPDE

$$u(t, x) - u(0, x) = \int_0^t u_{xx}(s, x)\, ds + w_t,$$

where w_t is a one-dimensional Wiener process. Hence comes the need of weights.

Next, equation (1.1) will be considered in $L_q((0,T), L_p)$-spaces, so that the powers of summability with respect to x and t are different.

Surprisingly enough, to the best of my knowledge, the case $q \neq p$ was never addressed before even for the heat equation in $\mathsf{R} \times \mathsf{R}^d$ without weights. We could only find references [1] and [12], where different powers of summability can be related to the Cauchy problem in $\{t > 0\}$ for $f = 0$. It turns out that in $\mathsf{R} \times \mathsf{R}^d$ the result we need can be obtained quite easily on the basis of a Banach space version of the Calderón-Zygmund theorem (see Sec. 2), which allows one to pass from $q = p$ to $q \neq p$.

For equation (1.1) in $\mathsf{R} \times \mathsf{R}^d_+$ we impose zero boundary condition and look for solutions in weighted Sobolev spaces with weights allowing the spatial derivatives of solutions to blow up near the boundary $x^1 = 0$. Our results in this setting extend the corresponding results in [5], where $q = p$.

This time we again use the Calderón-Zygmund theorem starting with the results valid for $q = p$. However, in order to check the conditions of this theorem, we need some nontrivial properties of the heat semigroup in weighted spaces, which we prove in Sec. 4. In Sec. 5, this allow us to get the result for $\mathsf{R} \times \mathsf{R}^d_+$, but only if $a^{1j} \equiv 0$ for $j \geq 2$. Usually, if one proves a sufficiently strong result for the heat equation, the same or very close result can be proved for equations with variable *continuous* coefficients. However, in the main application, which we have in mind, to stochastic partial differential equations from filtering theory, the regularity of a^{ij} in time is hard to control. Therefore, we always deal only with measurable coefficients and, in Sec. 6 after some additional work, we prove our main result for equations in $\mathsf{R} \times \mathsf{R}^d_+$ in full generality.

The arguments of Sec. 6 are based on Lemma 2.5, which also allows us to give a different proof of the result in $\mathsf{R} \times \mathsf{R}^d$ by using the Marcinkiewicz interpolation theorem rather than the Calderón-Zygmund theorem.

The proofs of part of the results are based on a general theorem saying, roughly speaking that, whatever estimate is true for the heat equation, it is also true for parabolic equations with coefficients depending only on time. This theorem is equally applicable to Sobolev and Hölder spaces.

Finally, it is worth noting that we also give results for the initial value problems. To give the reader a flavor of our results in $\mathsf{R} \times \mathsf{R}^d_+$ we state a particular case of Theorem 5.2.

Theorem 1.1. *Let* $p,q\in(1,\infty)$, $-1<\alpha<p-1$, $T\in(0,\infty)$, *and assume that we are given a function* $f(t,x)$ *defined for* $t>0, x\in\mathsf{R}^d_+$ *and such that*

$$\int_0^T \Big(\int_{\mathsf{R}^d_+}(x^1)^\alpha|x^1f(t,x)|^p\,dx\Big)^{q/p}dt<\infty.$$

Then on $[0,T]\times\bar{\mathsf{R}}^d_+$ *there is a unique function* u *satisfying the heat equation*

$$u_t=\Delta u+f\quad in\quad (0,T)\times\mathsf{R}^d_+,$$

vanishing for $t=0$ *and for* $x^1=0$ *in a natural sense and such that*

$$\int_0^T \Big(\int_{\mathsf{R}^d_+}(x^1)^\alpha[|u(t,x)/x^1|^p+|u_x(t,x)|^p+|x^1u_{xx}(t,x)|^p]\,dx\Big)^{q/p}dt<\infty.$$

2. MAIN RESULTS WITHOUT WEIGHTS

Here we consider equation (1.1) in usual Sobolev spaces. Theorem 2.1 is the main result of this section. One of its proofs is based on the Calderón-Zygmund theorem. It seems impossible to carry over this approach to the case of stochastic partial differential equations. Another proof uses the fact that

$$||u||_p^{np}=\int_{\mathsf{R}^{nd}}|u(x_1)\cdot...\cdot u(x_n)|^p\,dx_1\cdot...\cdot dx_n.$$

This allows us to reduce estimating the $L_{np}(L_p)$-norm of $u(t,x)$ to estimating L_p-norm of the function $u(t,x_1)\cdot\ldots\cdot u(t,x_n)$ which happens to satisfy the heat equation in a higher dimension space. It turns out that this device works equally well for stochastic partial differential equations.

Define

$$Lu = a^{ij}u_{x^ix^j} - u_t, \tag{2.1}$$

$$L_p = L_p(\mathsf{R}^d), \quad H^\gamma_p = (1-\Delta)^{-\gamma/2}L_p, \quad \mathbb{H}^{\gamma,q}_p = L_q(\mathsf{R}, H^\gamma_p),$$

$$\mathbb{H}^{\gamma,q}_p(T) = L_q((0,T), H^\gamma_p), \quad \mathbb{L}^q_p = \mathbb{H}^{0,q}_p, \quad \mathbb{L}^q_p(T) = \mathbb{H}^{0,q}_p(T).$$

It is well known that, for any $s\in\mathsf{R}$ and $f\in C^\infty_0(\mathsf{R}^d)$, there exists a unique bounded continuous function $u(t,x)$ on $[s,\infty)\times\mathsf{R}^d$ satisfying $Lu(t)=0$ for $t>s$ with initial condition $u(s)=f$. We denote

$$u(t) = T_{s,t}f$$

and recall that, for each s and t, the operator $T_{s,t}$ is written as the convolution of f with a Gaussian density. In particular, $T_{s,t}f$ is infinitely differentiable in x. Finally, for $f\in C^\infty_0(\mathsf{R}\times\mathsf{R}^d)$, let

$$Rf(t) := \int_{-\infty}^t T_{s,t}f(s)\,ds, \quad Af := D^2Rf.$$

Also remember that, for $f \in C_0^\infty(\mathsf{R} \times \mathsf{R}^d)$, the function Rf satisfies

$$LRf = -f.$$

Theorem 2.1. *Let $q, p \in (1, \infty)$, $\gamma \in \mathsf{R}$. Then the operator A is uniquely extendible to a bounded operator acting in $\mathbb{H}_p^{\gamma,q}$. If we keep the same notation for the extension, then*

$$||D^2 Rf||_{\mathbb{H}_p^{\gamma,q}} \leq N(\delta, d, q, p)||f||_{\mathbb{H}_p^{\gamma,q}}. \tag{2.2}$$

Let us emphasize that the constant N in (2.2) is independent of K (see (1.2)).

The following is a corollary of Theorem 2.1.

Theorem 2.2. *Let $q, p \in (1, \infty)$, $T \in (0, \infty)$, and $\gamma \in \mathsf{R}$. Take $\varepsilon > 0$, $f \in \mathbb{H}_p^{\gamma,q}(T)$ and $u_0 \in H_p^{\gamma+2-2/q+\varepsilon}$. Then in $\mathbb{H}_p^{\gamma+2,q}(T)$ there is a unique solution of (1.1) with the initial condition $u(0) = u_0$. For this solution*

$$||u_{xx}||_{\mathbb{H}_p^{\gamma,q}(T)} \leq N(||f||_{\mathbb{H}_p^{\gamma,q}(T)} + ||u_0||_{H_p^{\gamma+2-2/q+\varepsilon}}), \tag{2.3}$$

where $N = N(\delta, d, q, p, T, \varepsilon)$, and if $u_0 = 0$, then N is independent of T. Finally, if $q = p$, one can take $\varepsilon = 0$.

The following corollary of Theorem 2.2 is obtained by odd continuation of the functions involved.

Corollary 2.3. *All assertions of Theorem 2.2 hold true for* $\gamma = 0$ *if we replace* R^d *with* R^d_+ *everywhere, assume that* $a^{1j} \equiv 0$ *for* $j = 2, ..., d$, *and supplement equation (1.1) with zero boundary condition at* $x^1 = 0$.

To prove Theorem 2.1, we use the following Banach space version of the Calderón-Zygmund theorem. This a standard result which is discussed, for instance, in Chapter 1 of [10] and can be extracted from more general results of [1]. For a Hilbert space version of this theorem in the form of multipliers along with a version of Theorem 2.1 for $q = p$ and different operators L we refer the reader to [9].

Theorem 2.4. *Let* F *and* G *be Banach spaces,* $p \in (1, \infty)$, *and* $A : L_p(\mathsf{R}^n, F) \to L_p(\mathsf{R}^n, G)$ *be a linear bounded operator. Assume that if a bounded strongly measurable* F*-valued function* f *has compact support* Γ, *then, for almost any* $x \notin \Gamma$, *we have*

$$Af(x) = \int_{\mathsf{R}^n} K(x, y) f(y)\, dy,$$

where $K(x,y)$ is a bounded operator from F into G, defined for $x \neq y$, strongly measurable with respect to y with norm bounded in y outside any neighborhood of x. Also assume that $K(x,y)$ is strongly measurable with respect to x and there exists a constant N such that

$$\int_{|x-y|>2|y-z|} |K(x,y) - K(x,z)|\, dx \leq N$$

for any y and z, which holds, for instance, if $K(x,y)$ is weakly differentiable in y and

$$|\nabla_y K(x,y)| \leq N|x-y|^{-d-1}.$$

Then the operator A is uniquely extendible to a bounded operator from $L_q(\mathsf{R}^n, F)$ to $L_q(\mathsf{R}^n, G)$ for any $q \in (1,p]$ and A is of weak-type $(1,1)$ on bounded functions with compact support.

The first proof of Theorem 2.1. In [3] a general theorem is proved which roughly speaking says that, whatever estimate is true for the heat equation in translation invariant spaces, it is also true with the same constant for equation (1.1) with the coefficients depending only on t provided

$$(a^{ij}(t)) \geq (\delta^{ij}).$$

Therefore we may and will assume that $a^{ij} \equiv \delta^{ij}$. Also assuming $\gamma = 0$ does not restrict generality.

Now we are ready to use Theorem 2.4. From Sec. 4.3 of [8] we know that A is uniquely defined and is bounded as an operator acting in $L_p(\mathsf{R}, L_p)$. We are going to check that A satisfies the assumptions of Theorem 2.4 with $F = G = L_p$.

Observe the simple fact that, for $t > 0$, $k = 1, 2, ...$, and $f \in L_p$, we have

$$\partial T_t f/\partial t = \Delta T_t f \quad \text{and} \quad ||\partial^k T_t f/\partial t^k||_{L_p} \leq N t^{-k} ||f||_{L_p},$$

where N depends only on d and k. For $t > 0$ introduce the operator

$$K(t) = \Delta T_t : L_p \to L_p$$

with norm bounded by $N t^{-1}$, where N is independent of t. For $t \leq 0$, let $K(t) = 0$.

Since $a^{ij} = \delta^{ij}$, we have

$$Rf(t,x) = \int_{-\infty}^{t} T_{t-s} f(s, \cdot)(x)\, ds.$$

In addition, if t is at a distance from the support of f, then differentiating the above formula presents no difficulties and we find

$$Af(t,x) = \int_{-\infty}^{t} \Delta T_{t-s} f(s, \cdot)\, dx = \int_{\mathsf{R}} K(t-s) f(s)(x)\, ds.$$

In order to prove that the assumptions of Theorem 2.4 are satisfied, it only remains to use

$$||\partial K(t-s)f/\partial s||_{L_p} = ||\partial^2 T_{t-s}f/\partial s^2||_{L_p} I_{t>s} \leq N|t-s|^{-2}||f||_{L_p}.$$

By Theorem 2.4, A is well defined and bounded as an operator from $L_q(\mathsf{R}, L_p)$ into itself for $1 < q \leq p$. By considering the adjoint to A, we conclude that A is bounded in $L_q(\mathsf{R}, L_p)$ for any $q, p \in (1, \infty)$. The theorem is proved.

To give a different proof of Theorem 2.1 we prepare two auxiliary results. The first one is an equivalent restatement of the same basic apriori estimate used in the above proof of Theorem 2.1.

Lemma 2.5. *Let* $T \leq \infty$, $p \in (1, \infty)$, *and* $u \in L_p((0,T) \times \mathsf{R}^d) = \mathbb{L}_p^p(T)$ *be a solution of the equation* $Lu = f^{ij}_{x^i x^j}$ *with zero initial data and with* $f^{ij} \in L_p((0,T) \times \mathsf{R}^d)$. *Then*

$$||u||_{\mathbb{L}_p^p(T)} \leq N(d, \delta, p) \sum_{ij} ||f^{ij}||_{\mathbb{L}_p^p(T)}.$$

This lemma follows, for instance, from the results of Sec. 4.3 in [8] up to the fact that there the results are stated for the heat equation or from Theorem 5.1 of [4] up to the assertion that N is independent of T. The later is obtained in a standard way by using self similarity. In almost the same form as stated this lemma is proved in Appendix in [11].

In the next lemma we do the first step to considering the power of summability in t equal to multiples of p.

Lemma 2.6. *Let* $T \leq \infty$, $p \in (1, \infty)$, $n = 1, 2, \ldots$ *For* $k = 1, \ldots, n$, *let* $u^k \in \mathbb{H}^{2,p}_p(T)$ *be solutions of the equation*

$$u^k_t = a^{ij} u^k_{x^i x^j} + f^k$$

with zero initial data and with $f^k \in \mathbb{L}^p_p(T)$. *Then*

$$\int_0^T \prod_{k=1}^n ||\Delta u^k(t)||^p_{L_p} \, dt \leq N \sum_{k=1}^n \int_0^T ||f^k(t)||^p_{L_p} \prod_{j \neq k} ||\Delta u^j(t)||^p_{L_p} \, dt, \tag{2.4}$$

where $N = N(n, d, p, \delta)$.

Proof. Define $v^k = \Delta u^k$. For $X = (x_1, \ldots, x_n) \in \mathsf{R}^{nd}$ with $x_i \in \mathsf{R}^d$, define

$$V(t, X) = v^1(t, x_1) \cdot \ldots \cdot v^n(t, x_n).$$

Observe that

$$V_t(t, X) = \boldsymbol{L} V(t, X) + F(t, X),$$

where $\boldsymbol{L}V = a^{rs}(V_{x_1^r x_1^s} + ... + V_{x_n^r x_n^s})$,

$$F(t,X) = \Delta_{x_i} G^i(t,X), \quad G^i(t,X) = f^i(t,x_i) \prod_{j \neq i} v^j(t,x_j).$$

Hence by Lemma 2.5

$$||V||_{L_p((0,T)\times \mathsf{R}^{nd})} \leq N \sum_i ||G^i||_{L_p((0,T)\times \mathsf{R}^{nd})}$$

and this is exactly (2.4). The lemma is proved.

The second proof of Theorem 2.1. As in the first proof, we only have to consider the case $a^{ij} \equiv \delta^{ij}$ and $\gamma = 0$. Also obviously it suffices to prove (2.2) for $f \in C_0^\infty(\mathsf{R} \times \mathsf{R}^d)$. Without loss of generality we assume that $f(t) = 0$ for $t \leq 0$.

Let $u = Rf$. Then u is a classical solution of

$$u_t = \Delta u + f$$

for $t > 0$ with zero initial condition and, even more than that, $u(s) = 0$ for $s \leq 0$. In addition, it is easy to check that $u \in L_p((0,T), H_p^2)$ for each $T < \infty$.

Next we take $q = np$, where $n = 1, 2, ...$ By Lemma 2.6 applied to $u^k = u$ we have

$$||u_{xx}||^{np}_{L_{np}((0,T),L_p)} \leq N \int_0^T ||f(t)||^p_{L_p} ||u_{xx}(t)||^{(n-1)p}_{L_p}\, dt,$$

which by Hölder's inequality yields $||u_{xx}||_{L_{np}((0,T),L_p)} \leq N||f||_{L_{np}((0,T),L_p)}$. By letting $T \to \infty$ we obtain (2.2).

To treat general $q \geq p$, it suffices to use the Marcinkiewicz interpolation theorem. As in the first proof, the case $q \leq p$ is considered by duality. The theorem is proved.

3. Sobolev spaces with weights

M^α is the operator of multiplying by $(x^1)^\alpha$, $M = M^1$.

$\mathcal{D}(R^d_+)$ the space of all distributions on R^d_+.

$$L_{p,\theta} := H^0_{p,\theta} = L_p(\mathsf{R}^d_+, (x^1)^{\theta-d}\,dx).$$

If γ is a nonnegative integer, the space $H^\gamma_{p,\theta}$ is

$$\{u : u, x^1 u_x, \dots, (x^1)^{|\alpha|} D^\alpha u \in L_{p,\theta} \quad \forall \alpha : |\alpha| \leq \gamma\}.$$

For negative γ one may use duality and for all other γ the complex interpolation.

One may think that it is natural to replace $(x^1)^{|\alpha|}$ with $(x^1)^{\alpha_1}$...

Definition 3.1. Take and fix a nonnegative function $\zeta \in C_0^\infty(\mathsf{R}_+)$ such that

$$\sum_{n=-\infty}^{\infty} \zeta^p(e^{x-n}) \geq 1 \quad \forall x \in \mathsf{R}.$$

For $\gamma, \theta \in \mathsf{R},$ and $p \in (1, \infty)$ let $H_{p,\theta}^\gamma$ be the set of all distributions u on R_+^d such that

$$||u||_{H_{p,\theta}^\gamma}^p := \sum_{n=-\infty}^{\infty} e^{n\theta} ||u(e^n \cdot)\zeta||_{H_p^\gamma}^p < \infty.$$

Independence of ζ...

The space $C_0^\infty(\mathsf{R}_+^d)$ is dense in $H_{p,\theta}^\gamma$.

Theorem 3.2. (i) For any $a > 0$ and $\alpha \in \mathsf{R}$,

$$||u(a\,\cdot)||_{H_{p,\theta}^\gamma}^p \leq a^{-\theta} N ||u||_{H_{p,\theta}^\gamma}^p \leq N ||u(a\,\cdot)||_{H_{p,\theta}^\gamma}^p,$$

$$||M^\alpha u||_{H_{p,\theta}^\gamma} \leq N ||u||_{H_{p,\theta+p\alpha}^\gamma} \leq N ||M^\alpha u||_{H_{p,\theta}^\gamma},$$

$$||MDu||_{H_{p,\theta}^\gamma} + ||DMu||_{H_{p,\theta}^\gamma} \leq N ||u||_{H_{p,\theta}^{\gamma+1}}.$$

(ii) If $M^{-1}u \in H_{p,\theta}^\gamma$ and $\theta \neq d-1, d-1+p,$ then

$$||MD^2u||_{H_{p,\theta}^{\gamma-2}} \leq N ||M^{-1}u||_{H_{p,\theta}^\gamma} \leq N ||MD^2u||_{H_{p,\theta}^{\gamma-2}}. \tag{3.1}$$

(iii) Let $\mu \leq \gamma$ and $q \geq p$ be such that

$$\gamma - d/p = \mu - d/q.$$

Denote

$$\tau = \theta q/p$$

(so that $\tau/q = \theta/p$). Then for any $u \in H^{\gamma}_{p,\theta}$ we have

$$u \in H^{\mu}_{q,\tau}, \quad ||u||_{H^{\mu}_{q,\tau}} \leq N||u||_{H^{\gamma}_{p,\theta}}.$$

(iv) Assume $\gamma p > d$ and represent $\gamma - d/p$ as $k + \varepsilon$, where k is an integer and $\varepsilon \in (0,1]$. Let i, j be multi–indices such that $|i| \leq k, |j| = k$. Then for any $u \in H^{\gamma}_{p,\theta}$, we have

$$M^{|i|+\theta/p} D^i u \in C(\mathsf{R}^d_+), \quad M^{k+\varepsilon+\theta/p} D^j u \in \mathcal{C}^{\varepsilon}_{loc}(\mathsf{R}^d_+),$$

$$||M^{|i|+\theta/p} D^i u||_{C(\mathsf{R}^d_+)} \leq N||u||_{\gamma,p,\theta},$$

$$[M^{k+\varepsilon+\theta/p} D^j u]_{\mathcal{C}^{\varepsilon}(\mathsf{R}^d_+)} \leq N||u||_{\gamma,p,\theta},$$

where $\mathcal{C}^{\varepsilon}(\mathsf{R}^d_+)$ is the Zygmund space (coinciding with $C^{\varepsilon}(\mathsf{R}^d_+)$ if $\varepsilon \in (0,1)$).

Theorem 3.3. (i) We have

$$(H^{\gamma}_{p,\theta})' = H^{\gamma'}_{p',\theta'}, \quad \gamma' = -\gamma, \quad 1/p + 1/p' = 1, \quad \theta/p + \theta'/p' = d.$$

(ii) For $\kappa \in (0,1)$, $p \in (1,\infty)$, $\gamma_i, \theta_i \in \mathsf{R}$, $i = 0,1$,

$$\theta = \kappa\theta_1 + (1-\kappa)\theta_0, \quad \gamma = \kappa\gamma_1 + (1-\kappa)\gamma_0,$$

we have $[H^{\gamma_0}_{p,\theta_0}, H^{\gamma_1}_{p,\theta_1}]_\kappa = H^{\gamma}_{p,\theta}$.

For constant b, c define

$$\mathcal{L}_{b,c} = M^2\Delta + bMD_1 - c.$$

It turns out that the fractional powers of the degenerate operator $-\mathcal{L}_{b,c}$ can be defined and

$$H^{\gamma}_{p,\theta} = (-\mathcal{L}_{b,c})^{-\gamma/2} L_{p,\theta}.$$

Theorem 3.4. For any $\gamma, \nu, p, \theta, b$, there exist a constant $c_0 > 0$ such that, for any $c \geq c_0$, the operator

$$(-\mathcal{L}_{b,c})^{\gamma} : H^{2\gamma+\nu}_{p,\theta} \to H^{\nu}_{p,\theta}$$

is bounded and, for any $u \in H^{\gamma+\nu}_{p,\theta}$, we have

$$||u||_{2\gamma+\nu,p,\theta} \leq N||(-\mathcal{L}_{b,c})^{\gamma}u||_{\nu,p,\theta} \leq N||u||_{2\gamma+\nu,p,\theta}. \qquad (3.2)$$

The main difficulty of the theory is that Δ and $\mathcal{L}_{b,c}$ *do not commute*.

Theorem 3.5. For any b, γ, p, and θ, the operator $\mathcal{L}_{b,0}$ is the generator of an analytic semigroup T^b_t acting in $H^{\gamma}_{p,\theta}$.

Notice that, if $d = 1$, the theorem is quite simple. Indeed,

$$(D^2 + (b-1)D)(v(e^x)) = (\mathcal{L}_{b,0}v)(e^x),$$

so that the properties of the semigroup related to $\mathcal{L}_{b,0}$ can be easily obtained from the well-known properties of the semigroup related to the operator $D^2 + (b-1)D$ with constant coefficients. However, for $d \geq 2$, we do not know any easy way to deal with $\mathcal{L}_{b,0}$.

4. SOME SMOOTHING PROPERTIES OF THE HEAT SEMIGROUP IN SPACES $H^{\gamma}_{p,\theta}$

Let

$$D^2 = D^{2l}_{x^1} D^{2-2l}_{x'}$$

be a second order derivative operator, where $l \in \{0,1\}$, $D^{2l}_{x^1}$ is the operator of taking $2l$ derivative in x^1, and $D^{2-2l}_{x'}$ is an $(2-2l)$th derivative with respect to x'.

Denote by $\tilde{T}_t$ the semigroup associated with the operator Δ in R^d_+ with zero boundary condition on $\{x^1 = 0\}$.

The following is a corollary of a general theorem, which shows that the solutions of the Cauchy problem for equation (1.1) are "naturally smoother" than the initial data. This result interesting in its own right plays a central role in Sec. 5 in proving solvability of parabolic equations in weighted spaces.

For $t \geq s$ introduce the operator $\tilde{T}_{s,t}$ so that, for $f \in C^{\infty}_0(\mathsf{R}^d_+)$, $\tilde{T}_{s,t}f$ is the solution of the equation

$$u_t(t,x) = a^{ij}(t) u_{x^i x^j}(t,x)$$

for $t > s$ and $x \in \mathsf{R}^d_+$ satisfying $u(s,x) = f(x)$ with zero boundary condition at $x^1 = 0$. One has a very well known representation of the kernel of $\tilde{T}_{s,t}$ as the difference of certain Gaussian densities.

Corollary 4.1. *Assume* $a^{1j}(t) \equiv 0$ *for* $j = 2, ..., d$. *Let* $1 < p < \infty$, $\alpha, \theta, \gamma \in \mathsf{R}$,

$$d - 1 - 2p < \theta < d - 1 + p.$$

Then, for any $t > s$, *we have*

$$||MD_s D^2 \tilde{T}_{s,t} v||_{H^\gamma_{p,\theta}} \leq N(t-s)^{-2} ||Mv||_{H^\gamma_{p,\theta}},$$

with N *depending only on* $d, p, \theta, \gamma, m, K$, *and* δ.

5. EQUATION (1.1) IN R^d_+ IN SPACES WITH WEIGHTS

Remember that the spaces $H^\gamma_{p,\theta}$ and the operators $\tilde{T}_{s,t}$ are introduced in the beginning of Sec. 4 and before Corollary 4.1, respectively. Define

$$\tilde{R}f(t) = \int_{-\infty}^{t} \tilde{T}_{s,t} f(s)\,ds, \quad \tilde{A} = MD^2 \tilde{R} M^{-1} f.$$

Existence and uniqueness results for equation (1.1) in $(0, T) \times \mathsf{R}^d_+$ are based on the following counterpart of Theorem 2.1. In this section we discuss Theorem 5.1 under the additional assumption

$$a^{1j}(t) \equiv 0 \quad j = 2, ..., d. \tag{5.1}$$

postponing considering the general case until Sec. 6.

Theorem 5.1. *Let*

$$p, q \in (1, \infty), \quad \gamma \in \mathsf{R}, \quad d-1 < \theta < d-1+p.$$

Then the operator $\tilde{A}$ *is uniquely extendible to a bounded operator acting in* $L_q(\mathsf{R}, H^\gamma_{p,\theta})$. *If we keep the same notation for the extension, then*

$$||MD^2 \tilde{R} M^{-1} f||_{L_q(\mathsf{R}, H^\gamma_{p,\theta})} \leq N(\delta, d, q, p, \gamma) ||f||_{L_q(\mathsf{R}, H^\gamma_{p,\theta})}. \tag{5.2}$$

Proof. First we prove (5.2) for $q = p$ and $f \in C_0^\infty(\mathsf{R} \times \mathsf{R}^d)$. Without losing generality we assume that $f(s) = 0$ for $s \leq 0$. Then $\tilde{R} M^{-1} f$ is a classical solution of $Lu = -M^{-1} f$ vanishing for $t \leq 0$ and on $x^1 = 0$. Due to our restriction on θ and Theorem 5.6 of [5] we have that indeed (5.2) holds for $q = p$ and $f \in C_0^\infty(\mathsf{R} \times \mathsf{R}^d)$. By using the fact that $C_0^\infty(\mathsf{R} \times \mathsf{R}^d)$ is dense in $L_q(\mathsf{R}, H^\gamma_{p,\theta})$, we conclude that (5.2) holds for $q = p$ and any $f \in L_q(\mathsf{R}, H^\gamma_{p,\theta})$.

Below we also use Theorem 4.1 of [5], which says that

$$||M\Delta \,\cdot\, ||_{H^\gamma_{p,\theta}} \sim ||MD^2 \,\cdot\, ||_{H^\gamma_{p,\theta}}$$

in our range of θ. Therefore, instead of considering $\tilde{A}$ we may and will prove the theorem for

$$\bar{A} = M\Delta\tilde{R}M^{-1}f.$$

Notice that, if $f \in C_0^\infty(\mathsf{R} \times \mathsf{R}^d)$ and t is not in the support of $f(s)$ as a function of s, then $\tilde{T}_{s,t}M^{-1}f(s)$ is infinitely differentiable in x and

$$\bar{A}f(t) = M\Delta\tilde{R}M^{-1}f(t) = \int_{-\infty}^{t} M\Delta\tilde{T}_{s,t}M^{-1}f(s)\,ds. \tag{5.3}$$

Moreover, the boundedness of $\bar{A}$ as an operator from $L_p(\mathsf{R}, H^\gamma_{p,\theta})$ to $L_p(\mathsf{R}, H^\gamma_{p,\theta})$ and poinwise estimates of the operator $M\Delta\tilde{T}_{s,t}M^{-1}$ show that (5.3) holds for almost all t outside the support of f if f is a bounded $H^\gamma_{p,\theta}$-valued function with compact support. In other words, for those t,

$$\bar{A}f(t) = \int_{\mathsf{R}} K(t,s)f(s)\,ds,$$

where the operator $K(t,s)$ is defined by the formula

$$K(t,s)h = I_{t>s}M\Delta\tilde{T}_{s,t}M^{-1}h.$$

Observe that by a general theorem, $K(t,s)$ is a bounded operator from $H^\gamma_{p,\theta}$ into itself with norm less than $N|t-s|^{-1}$.

Now we claim that $\bar{A}$ is a bounded operator from $L_q(\mathsf{R}, H^\gamma_{p,\theta})$ to $L_q(\mathsf{R}, H^\gamma_{p,\theta})$ for $1 < q \leq p$. Clearly, we may assume that the coefficients a^{ij} are infinitely differentiable as long as we can prove that the estimates on $\bar{A}$ are independent of smoothness of a^{ij}. Then, owing to Theorem 2.4, to prove the claim, it suffices to show that the norm of $D_s K(t,s)$ as an operator in $H^\gamma_{p,\theta}$ is less than $N|t-s|^{-2}$ with N depending only on $d, p, q, \theta, \gamma, K$, and δ. But this is just the statement of Corollary 4.1 for $\alpha = m = 2$.

Thus, $\bar{A}$ is a bounded operator in $L_q(\mathsf{R}, H^\gamma_{p,\theta})$ for $1 < q \leq p$. The same is true for $1 < p \leq q$ which is proved by using duality and the fact that the dual to $H^\gamma_{p,\theta}$ is $H^{-\gamma}_{p',\theta'}$, with $1/p + 1/p' = 1$ and $\theta/p + \theta'/p' = d$, where θ' runs through $(d-1, d-1+p')$ as θ runs through $(d-1, d-1+p)$. The theorem is proved.

Now we can state our main result regarding the solvability of (1.1) in $(0,T) \times \mathsf{R}^d_+$ in weighted spaces. Denote

$$\mathbb{H}^{\gamma;q}_{p,\theta}(T) = L_q((0,T), H^\gamma_{p,\theta}), \quad \mathbb{H}^{\gamma;q}_{p,\theta} = L_q(\mathsf{R}, H^\gamma_{p,\theta}).$$

Remember that the operator L is introduced in (2.1).

Theorem 5.2. *Let* $p, q \in (1, \infty)$*,* $T \in (0, \infty)$*,* $\gamma \in \mathsf{R}$*,*

$$d-1<\theta<d-1+p, \quad \varepsilon>0.$$

$Mf \in \mathbb{H}^{\gamma,q}_{p,\theta}(T)$*, and* $M^{2/q-1-\varepsilon}u_0 \in H^{\gamma+2-2/q+\varepsilon}_{p,\theta}$*. Then in* $M\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)$ *there is a unique solution of equation (1.1) on* $(0, T)$ *with initial data* u_0*. For this solution*

$$||M^{-1}u||_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \le N_1||MLu||_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)} + N_2||M^{2/q-1-\varepsilon}u(0)||_{H^{\gamma+2-2/q+\varepsilon}_{p,\theta}}, \tag{5.4}$$

where $N_1 = N(d, p, q, \delta, K, \theta, \gamma)$ *and* $N_2 = N(d, p, q, \delta, K, \theta, \gamma, T)$*. In addition, if* $q = p$*, one can allow* $\varepsilon = 0$*, and then* N_2 *is independent of* T*.*

6. THEOREM 5.1 IN THE GENERAL CASE

We need two lemmas, in the first of which no restriction on θ is imposed. Remember that $Lu = a^{ij}u_{x^ix^j} - u_t$.

Lemma 6.1. *Let* $p \in (1, \infty)$*,* $n \in \{1, 2, ...\}$*,* $\gamma \ge \nu$*,* $\theta \in \mathsf{R}$*,* $M^{-1}u \in \mathbb{H}^{\nu,np}_{p,\theta}$*,* $Mf \in \mathbb{H}^{\gamma-2,np}_{p,\theta}$*. Assume* u *is a solution of* $Lu = f$ *in* $\mathsf{R} \times \mathsf{R}^d_+$*. Then* $M^{-1}u \in \mathbb{H}^{\gamma,np}_{p,\theta}$ *and*

$$||M^{-1}u||_{\mathbb{H}^{\gamma,np}_{p,\theta}} \le N(||MLu||_{\mathbb{H}^{\gamma-2,np}_{p,\theta}} + ||M^{-1}u||_{\mathbb{H}^{\nu,np}_{p,\theta}}), \tag{6.1}$$

where $N = N(d, n, p, \theta, \gamma, \nu, \delta)$*.*

Proof. Clearly (6.1) becomes stronger if ν decreases. Therefore we may assume that $\nu = \gamma - k$, where k is an integer, and bearing in mind an obvious induction, we see that, without loss of generality, we may let $\nu = \gamma - 1$.

Now notice that

$$||M^{-1}u||^{np}_{\mathbb{H}^{\gamma,np}_{p,\theta}} = \int_{\mathsf{R}} ||M^{-1}u(t)||^{np}_{H^{\gamma}_{p,\theta}}\,dt \le N \int_{\mathsf{R}} ||u(t)||^{np}_{H^{\gamma}_{p,\theta-p}}\,dt$$

$$= N \sum_{m_1,...,m_n=-\infty}^{\infty} e^{(\theta-p)\bar{m}} \int_{\mathsf{R}} \prod_{i=1}^{n} ||u(t, e^{m_i}\,\cdot)\zeta||^{p}_{H^{\gamma}_{p}}\,dt, \quad (6.2)$$

with $\bar{m} := m_1 + ... + m_n$. Here

$$||u(t, e^{m}\,\cdot)\zeta||^{p}_{H^{\gamma}_{p}} \le N||\Delta[u(t, e^{m}\,\cdot)\zeta]||^{p}_{H^{\gamma-2}_{p}}$$

$$= ||(1-\Delta)^{\gamma/2-1}\Delta[u(t, e^{m}\,\cdot)\zeta]||^{p}_{L_p}$$

$$= e^{m(p\gamma-d)}||(\lambda_m - \Delta)^{\gamma/2-1}\Delta[u(t)\zeta_m]||^{p}_{L_p},$$

where $\lambda_m = e^{-2m}$ and $\zeta_m(x) = \zeta(e^{-m}x)$. Furthermore, $L(u\zeta_m) = \bar{f}_m$, where,

$$\bar{f}_m = f\zeta_m + 2a^{ij}\zeta_{mx^i}u_{x^j} + ua^{11}\zeta_{mx^1x^1},$$

and similarly to the above computation

$$||(\lambda_m - \Delta)^{\gamma/2-1}\bar{f}_m(t)||^{p}_{L_p} = e^{-m(p\gamma-2p-d)}||\bar{f}_m(t, e^{m}\,\cdot)||^{p}_{H^{\gamma-2}_{p}}.$$

Therefore, by Lemma 2.6, which is obviously valid for R in place of $(0,T)$, for any $m_1,...,m_n$, we have

$$\int_{\mathsf{R}} \prod_{i=1}^{n} ||u(t,e^{m_i}\cdot)\zeta||^p_{H^\gamma_p}\,dt$$

$$\leq N\int_{\mathsf{R}} \sum_{i=1}^{n} e^{2m_i p}||\bar{f}_{m_i}(t,e^{m_i}\cdot)||^p_{H^{\gamma-2}_p} \prod_{j\neq i} ||u(t,e^{m_j}\cdot)\zeta||^p_{H^\gamma_p}\,dt.$$

Coming back to (6.2), we conclude

$$||M^{-1}u||^{np}_{\mathbb{H}^{\gamma,np}_{p,\theta}} \leq N\int_{\mathsf{R}} F(t)||u(t)||^{(n-1)p}_{H^\gamma_{p,\theta-p}}\,dt,$$

where

$$F(t) := \sum_{m=-\infty}^{\infty} e^{m(\theta+p)}||\bar{f}_m(t,e^m\cdot)||^p_{H^{\gamma-2}_p}.$$

Next we use (see [5]) that the operator M^β is a bounded operator from $H^\gamma_{p,\theta}$ to $H^\gamma_{p,\theta+\beta p}$ and that $M\nabla$ is a bounded operator from $H^\gamma_{p,\theta}$ to $H^{\gamma-1}_{p,\theta}$. Then we find

$$F(t) \leq N\sum_{m=-\infty}^{\infty} e^{m(\theta+p)}||f(t,e^m\cdot)\zeta||^p_{H^{\gamma-2}_p}$$

$$+N\sum_{m=-\infty}^{\infty} e^{m\theta}||u_x(t,e^m\cdot)\zeta'||^p_{H^{\gamma-2}_p} + N\sum_{m=-\infty}^{\infty} e^{m(\theta-p)}||u(t,e^m\cdot)\zeta''||^p_{H^{\gamma-2}_p}$$

$$\leq N(||Mf(t)||^p_{H^{\gamma-2}_{p,\theta}} + ||Mu_x(t)||^p_{H^{\gamma-2}_{p,\theta-p}} + ||M^{-1}u(t)||^p_{H^{\gamma-2}_{p,\theta}})$$

$$\leq N(||Mf(t)||^p_{H^{\gamma-2}_{p,\theta}} + ||u(t)||^p_{H^{\gamma-1}_{p,\theta-p}})$$

$$\leq N(||Mf(t)||^p_{H^{\gamma-2}_{p,\theta}} + ||M^{-1}u(t)||^p_{H^{\gamma-1}_{p,\theta}}).$$

Thus

$$||M^{-1}u||^{np}_{\mathbb{H}^{\gamma,np}_{p,\theta}}$$

$$\leq NE\int_{\mathsf{R}}(||Mf(t)||^p_{H^{\gamma-2}_{p,\theta}} + ||M^{-1}u(t)||^p_{H^{\gamma-1}_{p,\theta}})||M^{-1}u(t)||^{(n-1)p}_{H^{\gamma}_{p,\theta}}\,dt,$$

and, to get (6.1) for $\nu = \gamma - 1$, it only remains to use Hölder's inequality. The lemma is proved.

Lemma 6.2. *Let* $p, q \in (1, \infty)$, $d-1 < \theta < d-1+p$, $M^{-1}u \in \mathbb{H}^{2,q}_{p,\theta}$, *and* $Mf \in \mathbb{L}^q_{p,\theta}$. *Assume* u *is a solution of* $Lu = f$ *in* $\mathsf{R} \times \mathsf{R}^d_+$. *Then*

$$||M^{-1}u||_{\mathbb{L}^q_{p,\theta}} \leq N||MLu||_{\mathbb{L}^q_{p,\theta}}, \tag{6.3}$$

where $N = N(d, p, q, \theta)$.

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