Viscosity Solutions for Nonlinear Hyperbolic Systems

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Hyperbolic Systems of Conservation Laws

$$u_t + f(u)_x = 0$$

$u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ conserved quantities

$f = (f_1, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ fluxes

For gas-dynamics, conserved quantities are:

mass momentum energy

$$u_t + A(u)u_x = 0 \qquad\qquad A(u) = Df(u)$$

STRICTLY HYPERBOLIC: each matrix $A(u)$ has real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

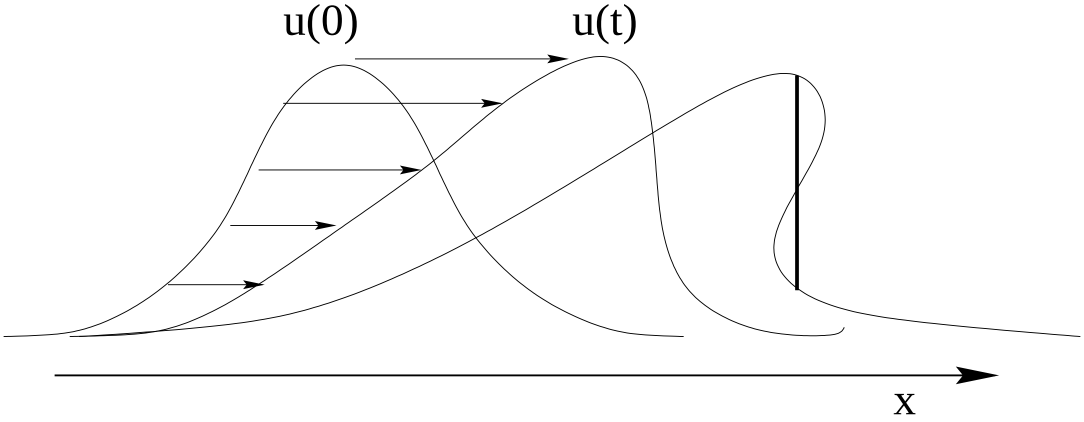
eigenvectors $r_1(u), \ldots, r_n(u)$

GENUINELY NONLINEAR: $D\lambda_i \cdot r_i > 0$

LINEARLY DEGENERATE: $D\lambda_i \cdot r_i \equiv 0$

Loss of Regularity

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$



Global solutions: in a space of discontinuous functions

$$u(t,\cdot) \in BV \quad \text{or} \quad \mathbf{L}^1_{loc}$$

Conservation equations interpreted in a distributional sense

$$\iint \left\{ u\phi_t + f(u)\phi_x \right\} dxdt \;=\; 0 \qquad\qquad \phi \in \mathcal{C}^1_c$$

Construction of Solutions

Standard techniques do not apply:

- Fixed point theorems
- Variational principles
- Upper, lower solutions

Two approaches:

The BV setting (J.Glimm, 1965)

- Approximate solutions are constructed in a class of piecewise constant functions
- Total variation is controlled by an Interaction Potential

The $\mathbf{L}^p$ setting (L.Tartar, R.DiPerna, 1979-1983)

- Consider vanishing viscosity approximations

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon\, u_{xx}^\varepsilon$$

- A bound in $\mathbf{L}^p$ yields a weakly converging sequence $u^\varepsilon \rightharpoonup u$ as $\varepsilon \to 0$
- By compensated compactness argument, u is a weak solution of the nonlinear system of conservation laws.
- Mainly restricted to 2×2 systems

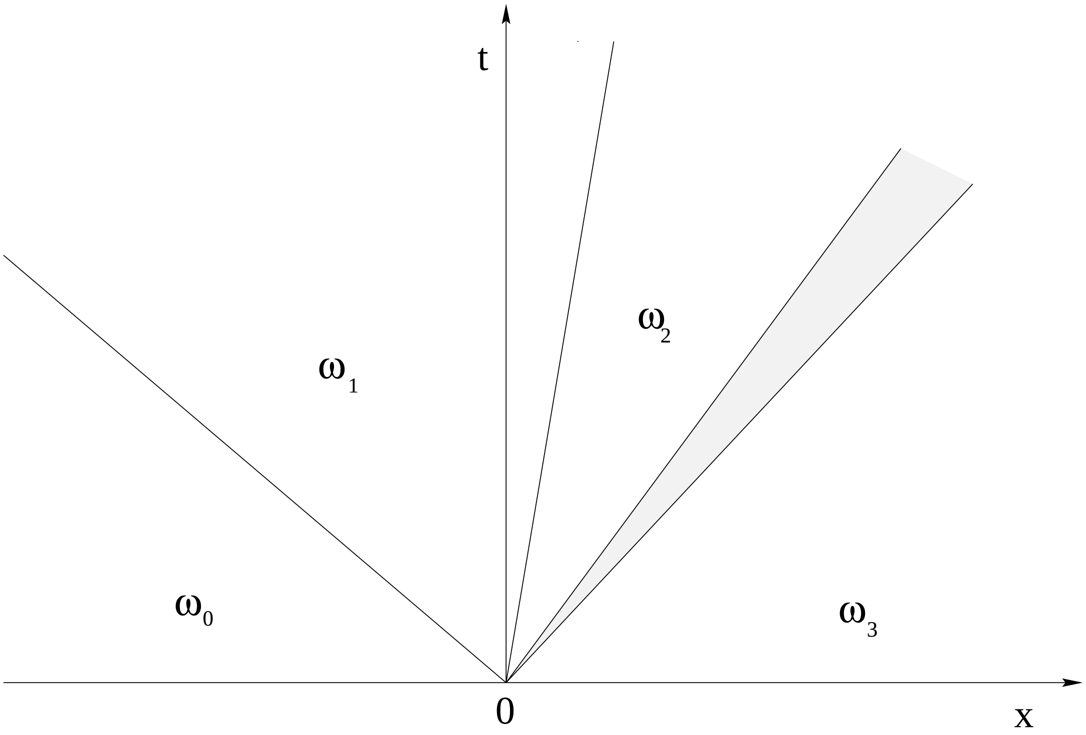
Construction of BV solutions

Building block: the Riemann Problem

$$u_t + f(u)_x = 0$$

$$u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

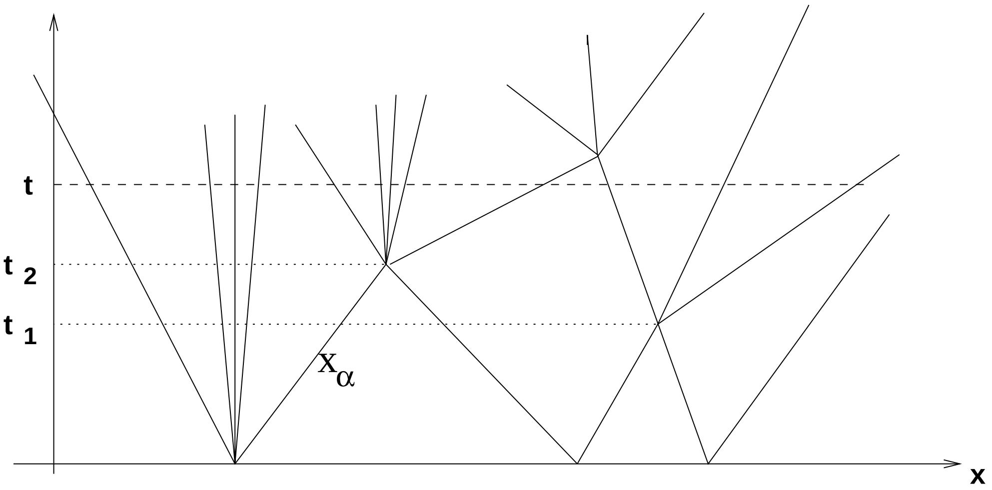
Self-similar solution (P. Lax, 1957)



The Cauchy problem

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)$$

Front Tracking Approximations



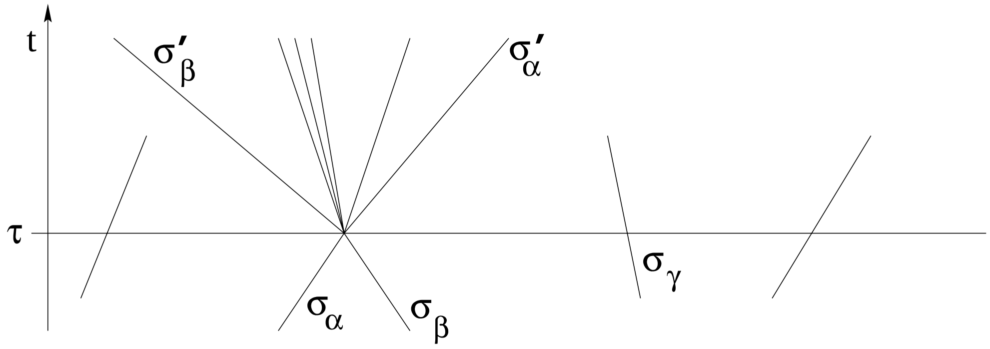
piecewise constant ε-approximate solution

$$\sum_\alpha \left| \dot{x}_\alpha (u_\alpha^+ - u_\alpha^-) - \big(f(u_\alpha^+) - f(u_\alpha^-) \big) \right| < \varepsilon \qquad \text{for a.e. } t > 0$$

Uniform BV bounds $\Longrightarrow$ compactness

$\Longrightarrow$ a subsequence converges in $\mathbf{L}^1_{loc}$ to a weak solution

Glimm interaction functional $\qquad Q(u) = \sum_{\text{approaching}} |\sigma_\alpha\, \sigma_\beta|$



Uniqueness and Continuous Dependence

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x) \tag{1}$$

Theorem 1 (Stability) *The solution of the Cauchy problem (1) obtained as limit of front tracking approximations is unique and depends Lipschitz continuously on the initial data.*

$$\big\|u(t,\cdot) - v(t,\cdot)\big\|_{\mathbf{L}^1} \le L \cdot \|\bar{u} - \bar{v}\|_{\mathbf{L}^1} \qquad\qquad t \ge 0$$

2×2	A.B. - R.M.Colombo (1994)
$n \times n$	A.B. - B.Piccoli (1996)
$n \times n$	A.B. - T.P.Liu - T.Yang (1998)

Lyapunov Functional

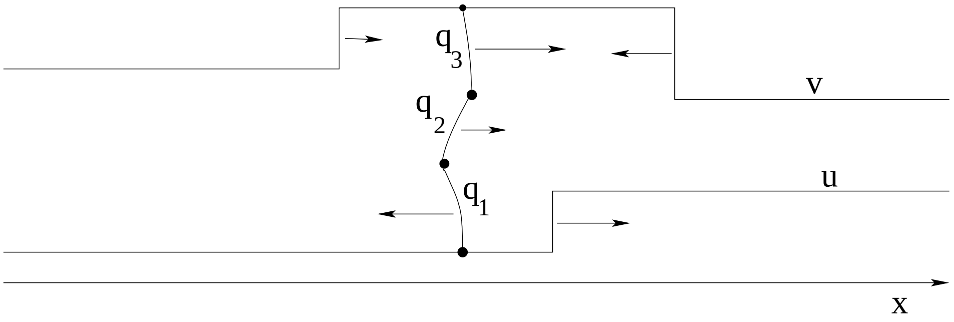
$$u_t + f(u)_x = 0$$

$$\frac{1}{C} \cdot \big\|v-u\big\|_{\mathbf{L}^1} \leq \Phi(u,v) \leq C \cdot \big\|v-u\big\|_{\mathbf{L}^1}$$

$$\frac{d}{dt}\Phi\big(u(t),\ v(t)\big) \leq 0$$

Construction of $\Phi(u,v)$ when u, v piecewise constant

$q_i(x) =$ i-th component of the jump $v(x) - u(x)$



$$\Phi(u,v) \doteq \sum_{i=1}^{n} \int_{-\infty}^{\infty} \big|q_i(x)\big| W_i(x)\ dx$$

$$\begin{aligned} W_i(x) &\doteq 1 + \kappa_1 \cdot \big[\text{total strength of waves in } u \text{ and in } v \text{ which approach } q_i(x)\big] \\ &\quad + \kappa_2 \cdot \big[\text{wave interaction potentials of } u \text{ and of } v\big] \\ &\doteq 1 + \kappa_1 A_i(x) + \kappa_2\big[Q(u) + Q(v)\big] \end{aligned}$$

Uniqueness

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x) \tag{1}$$

Semigroup of solutions obtained as limits of front-tracking approximations

$$S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D} \qquad\qquad t \mapsto u(t,\cdot) = S_t\bar{u}$$

$$\left\|S_t\bar{u} - S_t\bar{v}\right\|_{\mathbf{L}^1} \leq L \cdot \|\bar{u} - \bar{v}\|_{\mathbf{L}^1}$$

Theorem 2 (Uniqueness). *Let the map* $u = u(t,x)$ *be a weak solution of the Cauchy problem (1), with small total variation, which satisfies*

Lax Entropy Condition: *At every point of approximate jump, one has*

$$\lambda_i(u^+) \leq \lambda \leq \lambda_i(u^-)$$

Bounded Variation Condition: u *has finite total variation along every space-like curve* γ *in the* t*-*x *plane.*

Then $u(t,\cdot) = S_t\bar{u}$ *for all* $t \in [0,T]$, *hence the solution is unique.*

(A.B., P.LeFloch, P.Goatin, M.Lewicka, 1997-2000)

A. Bressan, *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem*, Oxford University Press 2000.

Vanishing Viscosity Approximations

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon\, u_{xx}^\varepsilon \qquad\qquad u(0,x) = \bar u(x) \tag{1}$$

Conjecture: as $\varepsilon \to 0$, the solution u^ε of (1) converges to the unique entropy weak solution of

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar u(x) \tag{2}$$

1. Scalar case (S.Kruzhkov, 1970)

- Based on comparison arguments
- Valid in several space dimensions.

2. Compensated Compactness (L.Tartar - R.DiPerna, 1979-83)

A subsequence u^ε converges (weakly) to some weak solution u of (2).

- applies mainly to 2×2 systems
- $u \in \mathbf{L}^\infty$, possibly $u \notin BV$. Uniqueness, stability ??

3. Singular Perturbations (J.Goodman - Z.Xin, 1992)

If u is piecewise smooth entropic solution of (2), then there exists a convergent sequence $u^\varepsilon \to u$.

- for $n \times n$ systems, piecewise smooth solutions are not dense

Key problem: derive BV bounds on $u^\varepsilon(t,\cdot)$, independent of ε. This will imply strong convergence $\|u^\varepsilon - u\|_{\mathbf{L}^1_{loc}} \to 0$ and uniqueness.

A priori BV bounds

The rescaling $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$ transforms

$$u_t + A(u)u_x = \varepsilon u_{xx} \qquad\qquad u(0,x) = \bar{u}(x)$$

into

$$u_t + A(u)u_x = u_{xx} \qquad\qquad u(0,x) = \bar{u}(\varepsilon x) \tag{3}$$

Assuming $\bar{u}$ small in BV, estimate: $\quad \text{Tot.Var.}\big\{u(t,\cdot)\big\} \quad t > 0$

Basic strategy:

1. Decompose the gradient along a suitable basis of unit vectors

$$u_x = \sum v^i \tilde{r}_i$$

2. Write an evolution equation for the gradient components

$$v^i_t + (\tilde{\lambda}_i v^i)_x - v^i_{xx} = \phi_i$$

3. Show that the source terms ϕ_i are INTEGRABLE, hence

$$\big\|v^i(t,\cdot)\big\|_{\mathbf{L}^1} \leq \big\|v^i(0,\cdot)\big\|_{\mathbf{L}^1} + \int_0^\infty \int_{I\!R} \big|\phi_i(s,x)\big|\, dxdt$$

4. Conclude that

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} = \big\|u_x(t,\cdot)\big\|_{\mathbf{L}^1} \leq \sum \big\|v^i(t,\cdot)\big\|_{\mathbf{L}^1} < \infty$$

Tracing Viscous Waves

$$u_t + A(u)u_x = u_{xx} \tag{3}$$

The decomposition

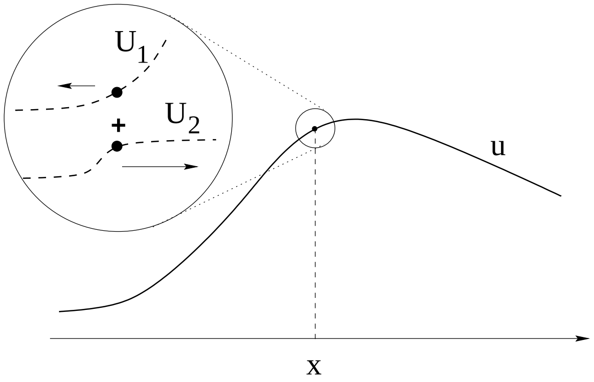
$$v^i = l_i \cdot u_x \qquad\qquad u_x = \sum_i v^i r_i \tag{4}$$

along the eigenvectors $r_1, \ldots, r_n$ of $A(u)$ yields

$$\begin{aligned} &v^i_t + (\lambda_i v^i)_x - v^i_{xx} \\ &= l_i \cdot \left\{ \sum_{j\neq k} \lambda_j [r_j, r_k] v^j v^k + 2 \sum_{j,k} (r_k \bullet r_j) v^j_x v^k + \sum_{j,k,\ell} [r_\ell,\ r_k \bullet r_j] v^j v^k v^\ell \right\} \end{aligned}$$

It does NOT work. One should achieve $\phi_i \equiv 0$ in case of a viscous travelling wave.

Problem: Given u, u_x, u_{xx} at a point x, is there a meaningful way to decompose u_x as a **sum of gradients of viscous travelling waves** ?



STEP 1: Construct a family of travelling wave profiles through a given state $u(x) \in \mathbb{R}^n$, having the right number of parameters to fit the data

$$(u_x,\, u_{xx}) \in \mathbb{R}^{n+n}$$

$$U_i(x) = u(x) \qquad\qquad i = 1, \ldots, n$$

$$\sum_i U_i'(x) = u_x(x)\,, \qquad\qquad \sum_i U_i''(x) = u_{xx}(x)$$

For each i, select a 2-parameter family of travelling waves

$$\big(A(U_i) - \sigma_i\big)U_i' = U_i''$$

A Center Manifold Approach

(Stefano Bianchini, 2000)

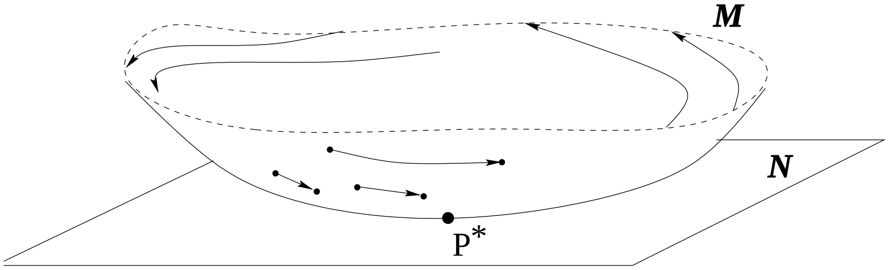
$$u_t + A(u)u_x = u_{xx}$$

travelling wave solution $\qquad u(t,x) = U(x - \sigma t)$

$$U'' = \big(A(U) - \sigma\big)U' \qquad \begin{cases} \dot u = v \\ \dot v = \big(A(u) - \sigma\big)v \\ \dot\sigma = 0 \end{cases} \tag{6}$$

Equilibrium point: $\quad P^* \doteq \big(u^*,\, 0,\, \lambda_i(u^*)\big)$

Center Manifold Theorem: In a neighborhood of the point P^* there exists a center manifold $\mathcal{M}_i$, of dimension $n+2$, locally invariant for the flow of (6).



For each $(u, v^i, \sigma_i) \approx \big(u^*,\ 0,\ \lambda_i(u^*)\big)$ this construction singles out an i-travelling wave $U(s)$ passing through u (say, at $s = 0$) such that

$$v^i = \pm\big|U'\big| \ = \ \text{signed strength}$$
$$\sigma_i \ = \ \text{speed}$$

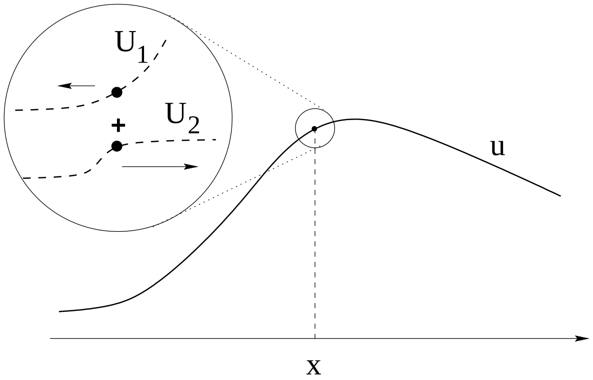
Call $\quad \tilde r_i(u, v^i, \sigma_i) = U'/|U'| \quad$ the unit vector tangent to this travelling wave profile

Wave Decomposition

$$\tilde{r}_i = \tilde{r}_i(u, v^i, \sigma_i) \qquad\qquad \text{is a smooth map}$$

When $v^i = 0$, one has $\tilde{r}_i(u, 0, \sigma_i) = r_i(u) = i$-eigenvector of $A(u)$, hence $\{\tilde{r}_1, \ldots, \tilde{r}_n\}$ is a basis of linearly independent unit vectors

Given a smooth function $u : \mathbb{R} \mapsto \mathbb{R}^n$, at each point x we seek a decomposition of u_x as a sum of gradients of travelling waves, depending on the second order jet (u, u_x, u_{xx})



Key idea: solve simultaneously the $2n$ equations for $v^1, \ldots v^n$, $w^1, \ldots, w^n$

$$\begin{aligned} u_x &= \sum_{i=1}^{n} v^i \tilde{r}_i(u, v^i, \sigma_i) \\ u_t \doteq u_{xx} - A(u)u_x &= \sum_{i=1}^{n} w^i \tilde{r}_i(u, v^i, \sigma_i) \end{aligned} \qquad \sigma_i = -\theta_i\left(\frac{w^i}{v^i}\right) \tag{8}$$

For a j-travelling wave contained in the center manifold, this yields

$$u_x = v^j \tilde{r}_j \qquad u_t = w^j \tilde{r}_j \qquad\qquad \sigma_j = -\frac{w^j}{v^j} = -\frac{u_t}{u_x}$$

$$v^i = 0 \qquad\qquad \text{for} \qquad i \neq j$$

Evolution of Gradient Components

Given u, u_x, u_{xx} , there exists a unique decomposition

$$u_x = \sum v^i \tilde r_i \qquad\qquad u_t = \sum w^i \tilde r_i$$

Defining $\tilde\lambda(u, v^i, \sigma_i) \doteq \langle \tilde r_i\,,\ A(u)\tilde r_i\rangle$, after n^{n^n} computations one finds

$$\begin{cases} v^i_t + (\tilde\lambda_i v^i)_x - v^i_{xx} = \phi_i \\ w^i_t + (\tilde\lambda_i w^i)_x - w^i_{xx} = \psi_i \end{cases} \tag{9}$$

$$\begin{aligned} \phi_i,\ \psi_i =& \mathcal{O}(1)\cdot \sum |w^j + \theta_j v^j| \cdot \big[|v^j w^j|/\delta_1 + |v^j_x| + |w^j_x|\big] && \textbf{(wrong speed)} \\ &+ \mathcal{O}(1)\cdot \sum |w^j_x v^j - v^j_x w^j| && \textbf{(change in strength)} \\ &+ \mathcal{O}(1)\cdot \sum \frac{1}{\delta_1} \left| v^j \left(\frac{w^j}{v^j}\right)_x \right|^2 && \textbf{(change in speed)} \\ &+ \mathcal{O}(1)\cdot \sum_{j\neq k} \big[|v^j v^k| + |v^j_x v^k| + |v^j w^k| + |v^j_x w^k| + |w^j w^k|\big] \\ & && \textbf{(interaction of waves of different families)} \end{aligned}$$

KEY ESTIMATES

Wrong speed $\Longrightarrow$ Energy estimates for the heat equation

Change in wave strength $\Longrightarrow$ Area functional

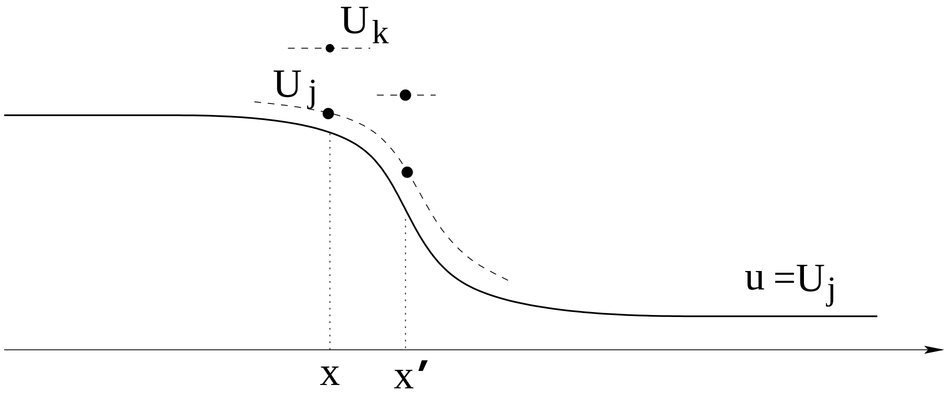
Change in wave speed $\Longrightarrow$ Curve shortening

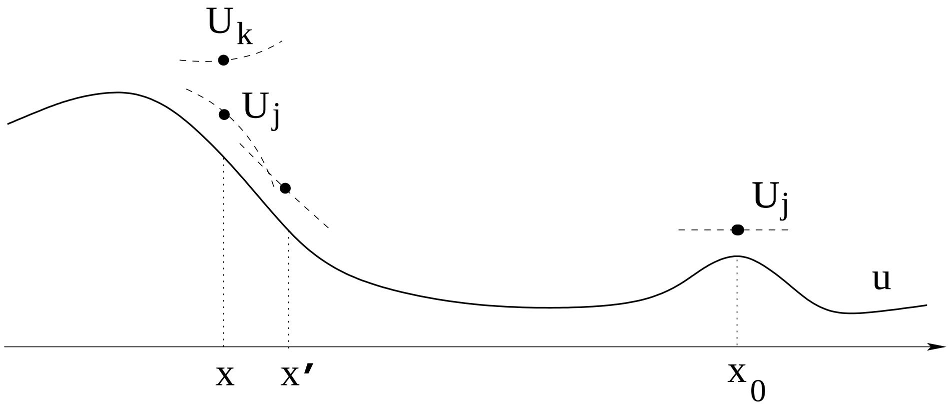
Interaction of waves of different families $\Longrightarrow$ Wave interaction potential

(A.B. and S.Bianchini, 1999-2000)

$$\big\|v^i(t,\cdot)\big\|_{\mathbf{L}^1}\,,\quad \big\|w^i(t,\cdot)\big\|_{\mathbf{L}^1}\,,\quad \big\|u_x(t,\cdot)\big\|_{\mathbf{L}^1} \;=\; \mathcal{O}(1) \qquad \text{for all } t \geq 0$$

Motivation of the Source Terms





Wrong speed: The cut-off function is active, near a point x_0 where $u_x \approx 0$, so that $|u_{xx}| >> |u_x|$

Change in wave speed, or strength: The viscous travelling j-wave that best approximates u near a point x is not the same at a nearby point x'

Transversal wave interactions: At a point x, waves of distinct families $j \neq k$ are present

Interaction of Viscous Waves of Distinct Families

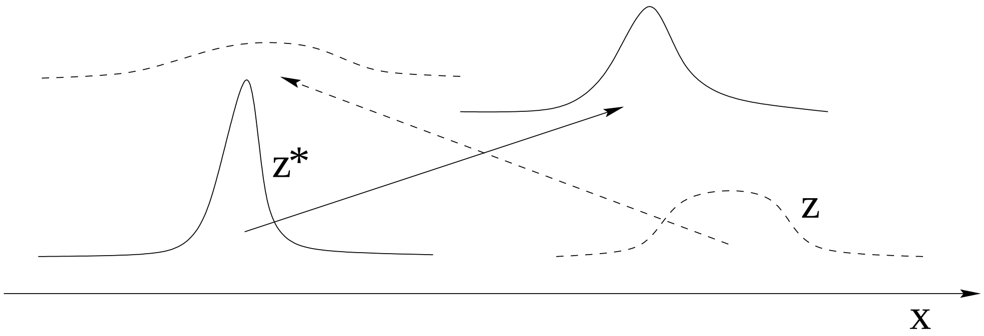
$$\begin{cases} z_t + \big[\lambda(t,x)z\big]_x - z_{xx} = 0 \\ z^*_t + \big[\lambda^*(t,x)z^*\big]_x - z^*_{xx} = 0 \end{cases}$$

$$\inf_{t,x} \lambda^*(t,x) - \sup_{t,x} \lambda(t,x) \geq c$$

$z \doteq$ density of waves with speed λ
$z^* \doteq$ density of waves with speed λ^*

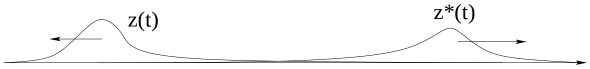
Total amount of interaction:

$$\int_0^\infty \int_{\mathbb{R}} \big|z(t,x)\big| \cdot \big|z^*(t,x)\big|\, dxdt$$

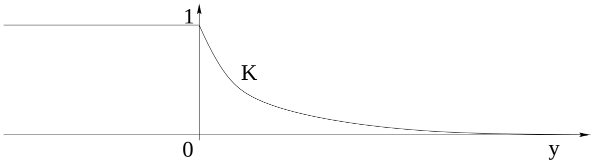


A Potential for Transversal Wave Interactions

$$Q(z,z^*) \doteq \iint_{I\!R^2} K(x_2-x_1)\big|z(x_1)\big|\,\big|z^*(x_2)\big|\,dx_1dx_2$$



$$K(y) \doteq \begin{cases} e^{-cy/2} & \text{if} \quad y>0 \\ 1 & \text{if} \quad y\le 0 \end{cases}$$



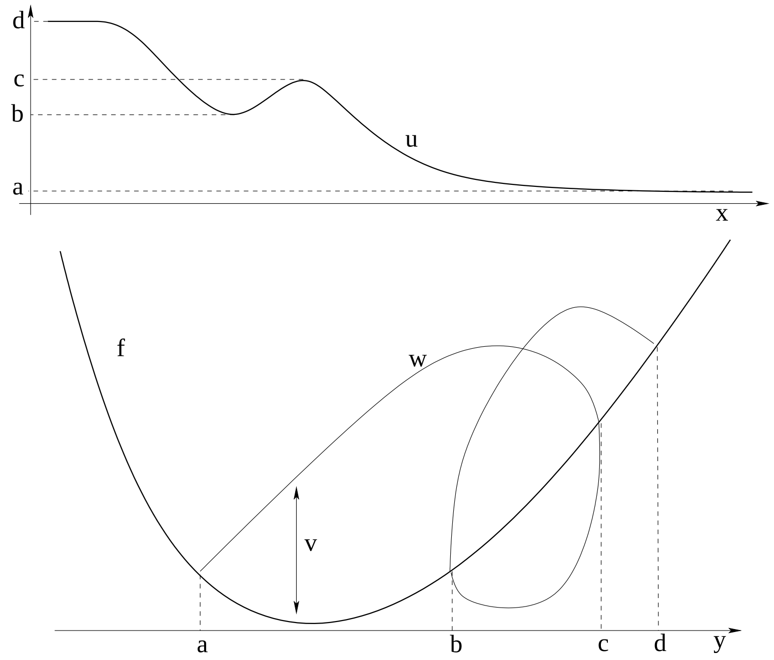
$$\begin{aligned}\int_0^\infty \int_{I\!R} \big|z(t,x)\big|\,\big|z^*(t,x)\big|\,dxdt &\le -\int_0^\infty \left[\frac{d}{dt}\frac{Q(t)}{c}\right]\,dt \\ &\le \frac{Q(0)}{c} \\ &\le \frac{1}{c}\int_{I\!R}\big|z(0,x)\big|\,dx\cdot\int_{I\!R}\big|z^*(0,x)\big|\,dx\end{aligned}$$

Lyapunov Functionals for the Scalar Viscous Conservation Law

$$u_t + f(u)_x = u_{xx}$$

The curve $$\gamma \doteq \begin{pmatrix} u \\ f(u) - u_x \end{pmatrix} = \begin{pmatrix} \text{conserved quantity} \\ \text{flux} \end{pmatrix}$$

evolves according to $$\gamma_t + f'(u)\gamma_x = \gamma_{xx}$$



Curve Length

$$\gamma \doteq \begin{pmatrix} u \\ f(u) - u_x \end{pmatrix} \qquad\qquad \gamma_x = \begin{pmatrix} v \\ w \end{pmatrix} \doteq \begin{pmatrix} u_x \\ -u_t \end{pmatrix}$$

Length $$L(\gamma) \doteq \int |\gamma_x|\, dx = \int \sqrt{v^2 + w^2}\, dx$$

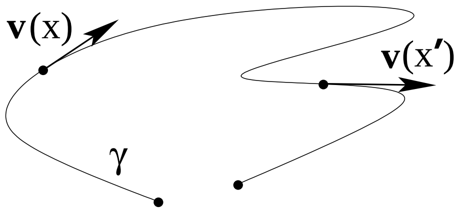
$$\gamma_t + f'(u)\gamma_x = \gamma_{xx} \qquad \Longrightarrow \qquad \frac{dL}{dt} \leq 0$$

$$\begin{aligned} -\frac{d}{dt} L\big(\gamma(t)\big) &= \int_{\mathbb{R}} \frac{|v|\,\big[(w/v)_x\big]^2}{\big(1 + (w/v)^2\big)^{3/2}}\, dx \\ &\geq \frac{1}{(1+\delta^2)^{3/2}} \int_{|w/v| \leq \delta} |v|\,\big[(w/v)_x\big]^2\, dx \end{aligned}$$

$$\begin{aligned} \int_0^\infty \int_{|w/v|\leq\delta} |v|\,\big[(w/v)_x\big]^2\, dx\, dt &= \mathcal{O}(1) \cdot \int_0^\infty \left| \frac{d}{dt} L\big(\gamma(t)\big) \right| dt \\ &= \mathcal{O}(1) \cdot L\big(\gamma(0)\big) \end{aligned}$$

Area Swept by a Curve

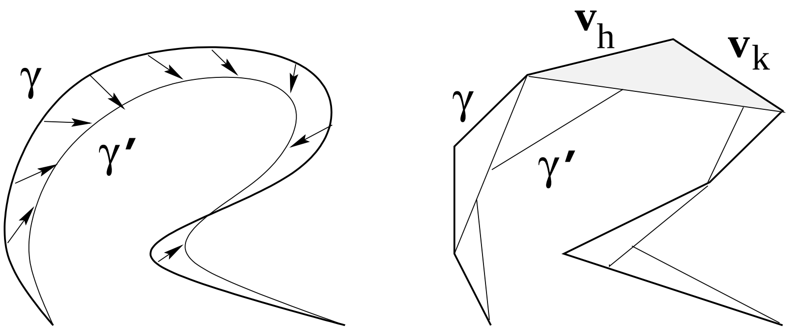
$$\gamma_t + f'(u)\gamma_x = \gamma_{xx}$$



Area functional $$Q(\gamma) \doteq \frac{1}{2}\iint_{x<x'} \big|\gamma_x(x)\wedge\gamma_x(x')\big|\,dx\,dx'$$

controls the area swept by the curve γ: $|dA| \le -dQ$

(for a proof, think of polygonal approximations)



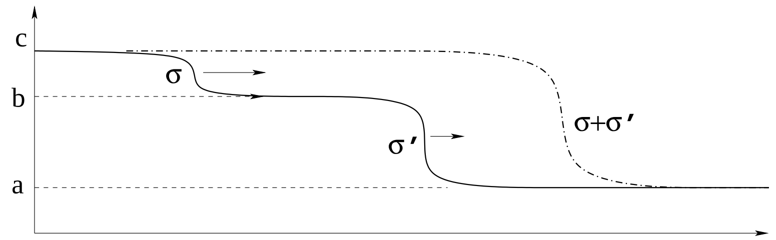
$$Q(\gamma) = \frac{1}{2}\sum_{i<j}\big|\mathbf{v}_i\wedge\mathbf{v}_j\big| \qquad |dA| = \frac{1}{2}\big|\mathbf{v}_h\wedge\mathbf{v}_k\big| \le -dQ$$

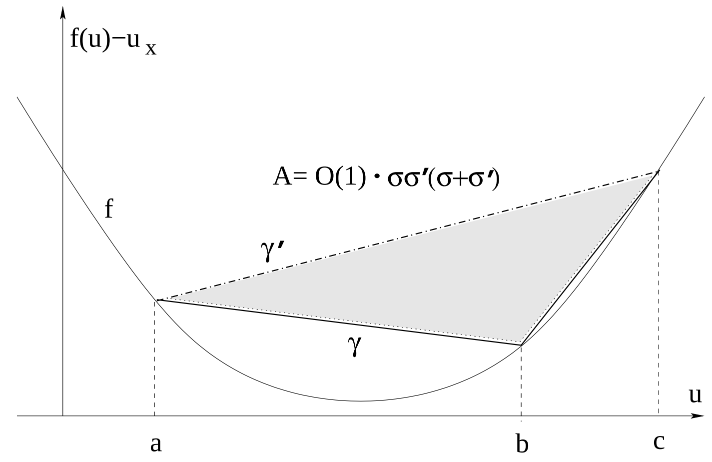
$$\gamma_t + f'(u)\gamma_x = \gamma_{xx} \qquad\qquad \gamma_x = \begin{pmatrix} v \\ w \end{pmatrix} \doteq \begin{pmatrix} u_x \\ -u_t \end{pmatrix}$$

$$-\frac{dQ}{dt} \geq \left|\frac{dA}{dt}\right| = \int |\gamma_t \wedge \gamma_x|\, dx = \int |\gamma_{xx} \wedge \gamma_x|\, dx = \int |v_x w - v w_x|\, dx$$

$$\int_0^\infty \int |v_x w - v w_x|\, dx\, dt \ \leq\ \int_0^\infty \left|\frac{dQ\big(\gamma(t)\big)}{dt}\right| dt \ \leq\ Q\big(\gamma(0)\big)$$

Genuinely Nonlinear Case





Theorem (S. Bianchini, 2001). Consider a strictly hyperbolic system with viscosity

$$u_t + A(u)u_x = \varepsilon\, u_{xx} \qquad\qquad u(0,x) = \bar u(x)\,. \tag{10}$$

There exists $\delta > 0$ such that the following holds. If

$$\text{Tot.Var.}\{\bar u\} < \delta\,, \qquad\qquad \|\bar u\|_{\mathbf{L}^\infty} < \delta\,,$$

then the Cauchy problem (10) admits a unique solution $u^\varepsilon(t,\cdot) = S_t^\varepsilon \bar u$, defined for all $t \geq 0$. Moreover, we have the estimates

$$\text{Tot.Var.}\big\{S_t^\varepsilon \bar u\big\} \leq C\, \text{Tot.Var.}\{\bar u\}\,, \tag{\textbf{BV bounds}}$$

$$\big\|S_t^\varepsilon \bar u - S_t^\varepsilon \bar v\big\|_{\mathbf{L}^1} \leq L\, \|\bar u - \bar v\|_{\mathbf{L}^1} \tag{$\mathbf{L^1}$ \textbf{stability}}$$

(Convergence) As $\varepsilon \to 0$, the solutions u^ε converge to the trajectories of a semigroup S, such that

$$\big\|S_t \bar u - S_t \bar v\big\|_{\mathbf{L}^1} \leq L\, \|\bar u - \bar v\|_{\mathbf{L}^1} \qquad\qquad t \geq 0\,.$$

These vanishing viscosity limits can be regarded as the unique **viscosity solutions** of the hyperbolic Cauchy problem

$$u_t + A(u)u_x = 0 \qquad\qquad u(0,x) = \bar u(x)\,.$$

- In the conservative case $A(u) = Df(u)$, the viscosity solutions are weak solutions to
$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar u(x)$$
satisfying the Liu entropy conditions.

- In the case where each field is genuinely nonlinear or linearly degenerate, the viscosity solutions coincide with the unique limits of Glimm or front tracking approximations.

Concluding Remarks

- Alternative approaches to systems of conservation laws:

  1. The BV setting (Glimm). Control of the total variation achieved by an interaction potential.

  2. The $\mathbf{L}^p$ setting (DiPerna). Convergence of approximate solutions achieved by compensated compactness.

\- In the BV setting the Cauchy problem is well posed. Examples show it can be ill posed for $\mathbf{L}^\infty$ data.

\- Convergence of vanishing viscosity solutions can also be achieved in BV, controlling the total variation by suitable Lyapunov functionals.

- Riemann Problem (and interactions of elementary waves) is no longer at the center stage.

\- Greater generality (no assumptions about genuine nonlinearity).

\- Reduce the gap between analytical theory and numerical algorithms.